Ramsey theory and the geometry of Banach spaces

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1.a. The Hales–Jewett theorem

The following result is due to Hales & Jewett (1963), and the corresponding bounds are due to Shelah (1988).

Theorem

For every pair k, r of positive integers with $k \geqslant 2$ there exists a positive integer N with the following property. If $n \geqslant N$, then for every alphabet A with $|A| = k$ and every r-coloring of A^n there exists a variable word w over A of length n such that the set $\{w(a) : a \in A\}$ is monochromatic. The least positive integer with this property is denoted by $\mathrm{HJ}(k, r)$.

Moreover, the numbers $\mathrm{HJ}(k, r)$ are upper bounded by a primitive recursive function belonging to the class $\mathcal{E}^5$.

1.b. The Hales–Jewett theorem

Shelah's proof proceeds by induction on the cardinality of the finite alphabet A.

The general inductive step splits into two parts. First, given a finite coloring c of A^n, one finds a "subspace" W of A^n of large dimension such that the coloring c restricted on W is "simple". Once the coloring has been made "simple", the proof is completed with an appropriate application of the inductive assumptions.

Of course, to implement this strategy, one has to define what a "simple" coloring actually is. We will come up again on this issue later on.

2.a. Colorings of combinatorial spaces

Let A be a finite alphabet with $|A| \geqslant 2$ and $d \geqslant 1$.

A d-**dimensional combinatorial space** of $A^{<\mathbb{N}}$ is a set of the form

$$\{w_0(a_0)^\frown \dots ^\frown w_{d-1}(a_{d-1}) : a_0, \dots, a_{d-1} \in A\}$$

where $w_0, \dots, w_{d-1}$ are variable words over A. (Note that for every combinatorial space W of $A^{<\mathbb{N}}$ there exists a unique positive integer n such that $W \subseteq A^n$.) A 1-dimensional combinatorial space is called a **combinatorial line**.

For every combinatorial space W of $A^{<\mathbb{N}}$ and every positive integer $m \leqslant \dim(W)$ by $\mathrm{Subsp}_m(W)$ we denote the set of all m-dimensional combinatorial spaces of $A^{<\mathbb{N}}$ which are contained in W.

2.b. Colorings of combinatorial spaces

The following result is a variant of the Graham–Rothschild theorem (1971). The corresponding bounds are essentially due to Shelah (1988).

Theorem

For every quadruple k, d, m, r of positive integers with $k \geqslant 2$ and $d \geqslant m$ there exists a positive integer N with the following property. If $n \geqslant N$ and A is an alphabet with $|A| = k$, then for every n-dimensional combinatorial space W of $A^{<\mathbb{N}}$ and every r-coloring of $\mathrm{Subsp}_m(W)$ there exists $V \in \mathrm{Subsp}_d(W)$ such that the set $\mathrm{Subsp}_m(V)$ is monochromatic. The least positive integer with this property is denoted by $\mathrm{GR}(k, d, m, r)$.

Moreover, the numbers $\mathrm{GR}(k, d, m, r)$ are upper bounded by a primitive recursive function belonging to the class $\mathcal{E}^6$.

3. The Carlson–Simpson theorem

Theorem (Carlson & Simpson – 1984)

For every finite alphabet A with $|A| \geqslant 2$ and every finite coloring of the set of all words over A there exist a word w over A and a sequence (u_n) of left variable words over A such that the set

$$\{w\} \cup \{w^\frown u_0(a_0)^\frown \dots ^\frown u_n(a_n) : n \in \mathbb{N} \text{ and } a_0, \dots, a_n \in A\}$$

is monochromatic.

The Carlson–Simpson Theorem is not only an infinite-dimensional extension of the Hales–Jewett theorem, but also refines the Hales–Jewett theorem by providing information on the structure of the wildcard set of the monochromatic variable word.

4.a. Carlson's theorem

Let A be a finite alphabet with $|A| \geqslant 2$ and $\mathbf{w} = (w_n)$ a sequence of variable word over A.

An **extracted variable word** of (w_n) is a variable word over A of the form

$$w_{i_0}(\alpha_0)^\frown \dots {}^\frown w_{i_n}(\alpha_n)$$

where $n \in \mathbb{N}$, $i_0 < \dots < i_n$ and $\alpha_0, \dots, \alpha_n \in A \cup \{x\}$. (Note that there exists $i \in \{0, \dots, n\}$ such that $\alpha_i = x$.)

By $\mathrm{EV}[\mathbf{w}]$ we denote the set of all extracted variable words of $\mathbf{w}$.

4.b. Carlson's theorem

Theorem (Carlson – 1988)

Let A be a finite alphabet with $|A| \geqslant 2$ and $\mathbf{w} = (w_n)$ a sequence of variable words over A. Then for every finite coloring of $\mathrm{EV}[\mathbf{w}]$ there exists an extracted subsequence $\mathbf{v} = (v_n)$ of $\mathbf{w}$ such that the set $\mathrm{EV}[\mathbf{v}]$ is monochromatic.

Carlson's theorem is one of the finest results in Ramsey theory. It unifies and extends several results, including the Carlson–Simpson theorem, Hindman's theorem and many more.

5.a. The density Hales–Jewett theorem

The following fundamental result of Ramsey theory is known as the *density Hales–Jewett theorem*.

Theorem (Furstenberg & Katznelson – 1991)

For every integer $k \geqslant 2$ and every $0 < \delta \leqslant 1$ there exists a positive integer N with the following property. If $n \geqslant N$ and A is an alphabet with $|A| = k$, then every $D \subseteq A^n$ with $|D| \geqslant \delta |A^n|$ contains a combinatorial line of A^n. The least positive integer N with this property is denoted by $\mathrm{DHJ}(k, \delta)$.

The best known upper bounds for the numbers $\mathrm{DHJ}(k, \delta)$ have an Ackermann-type dependence with respect to k. (Polymath, 2009 — D, Kanellopoulos & Tyros, 2012).

It is a central open problem to decide whether the numbers $\mathrm{DHJ}(k, \delta)$ are upper bounded by a primitive recursive function.

5.b. The density Hales–Jewett theorem

The density Hales–Jewett theorem has a number of consequences, including:

- Szemerédi's theorem (1975);
- the multidimensional Szemerédi theorem (Furstenberg & Katznelson, 1978);
- the density version of the affine Ramsey theorem (Furstenberg & Katznelson, 1985);
- Szemerédi's theorem for abelian groups (Furstenberg & Katznelson, 1985);
- the IP_r-Szemerédi theorem (Furstenberg & Katznelson, 1985).

6.a. The density Carlson–Simpson theorem

Theorem (D, Kanellopoulos & Tyros – 2012)

For every finite alphabet A with $|A| \geqslant 2$ and every set D of words over A satisfying

$$\limsup_{n\to\infty} \frac{|D \cap A^n|}{|A^n|} > 0$$

there exist a word w over A and a sequence (u_n) of left variable words over A such that the set

$$\{w\} \cup \{w^\frown u_0(a_0)^\frown \dots ^\frown u_n(a_n) : n \in \mathbb{N} \text{ and } a_0, \dots, a_n \in A\}$$

is contained in D.

6.b. The density Carlson–Simpson theorem

The proof is based on the following finite version.

Theorem (D, Kanellopoulos & Tyros – 2012)

For every pair k, m of positive integers with $k \geqslant 2$ and every $0 < \delta \leqslant 1$ there exists a positive integer N with the following property. If A is an alphabet with $|A| = k$, L is a finite subset of $\mathbb{N}$ of cardinality at least N and D is a set of words over A satisfying $|D \cap A^n| \geqslant \delta|A^n|$ for every $n \in L$, then there exist a word w over A and a finite sequence $(u_n)_{n=0}^{m-1}$ of left variable words over A such that the set

$$\{w\} \cup \{w^\frown u_0(a_0)^\frown \dots ^\frown w_n(a_n) : n < m \text{ and } a_0, \dots, a_n \in A\}$$

is contained in D. The least positive integer with this property is denoted by $\mathrm{DCS}(k, m, \delta)$.

6.c. The density Carlson–Simpson theorem

The main point is that the result is independent of the position of the finite set L. This is a strong structural property which does not follow from the corresponding infinite version with standard arguments based on compactness.

We also note that

$$\mathrm{DHJ}(k,\delta) \leqslant \mathrm{DCS}(k,1,\delta).$$

The proof is effective and yields explicit upper bounds for the numbers $\mathrm{DCS}(k,m,\delta)$. However, these upper bounds also have an Ackermann-type dependence with respect to k.

7. Probabilistic versions

The **probabilistic version** of a density result asserts that a dense set of a discrete structure not only will contain a substructure of a certain kind (arithmetic progression, combinatorial line, Carlson–Simpson space, etc.) but actually a non-trivial portion of them.

7.a. Probabilistic versions: Varnavides' theorem

A typical example is the following probabilistic version of Szemerédi's theorem, essentially due to Varnavides (1959).

For every integer $k \geqslant 2$ and every $0 < \delta \leqslant 1$ there exists a strictly positive constant $c(k, \delta)$ with the following property. If $n > c(k, \delta)^{-1}$, then every $D \subseteq [n]$ with $|D| \geqslant \delta n$ contains at least $c(k, \delta) n^2$ arithmetic progressions of length k.

(Here, $[n] := \{1, \ldots, n\}$.) The problem of obtaining good estimates for the constant $c(k, \delta)$ is of fundamental importance.

7.b. Probabilistic versions: Erdős & Simonovits (1983) – supersaturation

A similar phenomenon occurs in the context of graphs and uniform hypergraphs.

- For every positive integer n there exists a graph G on n vertices with $|E(G)| = \lfloor n^2/4 \rfloor$ and not containing a triangle, that is, a copy of K_3^2 (the complete graph on 3 vertices).
- On the other hand, if $|E(G)| > n^2/4$, then G contains a triangle (Mantel's theorem, 1907).
- Moreover, if $|E(G)| = c\binom{n}{2}$, then G contains at least $c(2c-1)\binom{n}{3} + o(n^3)$ triangles (Goodman).

It is a famous open problem (*hypergraph Turán problem*) to compute the critical threshold for K_t^r (the complete r-uniform hypergraph on t vertices) for any $t > r > 2$.

7.c. Probabilistic versions: balanced words

Contrary to what happens for the previous structures, there is no probabilistic version of the density Hales–Jewett theorem.

Example

A nonempty word $w = (w_0, \ldots, w_{n-1})$ over a finite alphabet A is called **balanced** if for every $a \in A$ we have

$$\left| |\{i \in \{0, \ldots, n-1\} : w_i = a\}| - \frac{n}{|A|} \right| \leqslant n^{2/3}.$$

Then we have $\mathbb{P}_{A^n}(\mathrm{Balanced}) = 1 - o(1)$ but

$$\mathbb{P}\big(\{L \in \mathrm{Lines}(A^n) : L \subseteq \mathrm{Balanced}\}\big) = o(1).$$

(Here, all measures are uniform probability measures.)

7.d. Probabilistic versions: density Hales–Jewett theorem (cont'd)

In spite of the previous example, it turns out that dense subsets of hypercubes indeed contain plenty of combinatorial lines, but when restricted on appropriately chosen combinatorial spaces. In other words, there is a “local” probabilistic version of the density Hales–Jewett theorem. This information is one of the key components of all known combinatorial proofs of the density Hales–Jewett theorem.

The method developed in order to obtain this “local” probabilistic version is quite general and works for most combinatorial structures of interest (including Carlson–Simpson spaces, polynomial spaces and many more).

8. Pseudorandomness

The **regularity method** is a remarkable discovery of Szemerédi asserting that dense sets of discrete structures are inherently pseudorandom. The method was first developed in the context of graphs, but it was realized recently that it can be formulated as an abstract probabilistic principle.

Convention: all probability spaces will be standard Borel.

8.a. Pseudorandomness: martingale difference sequences

A **martingale difference sequence** is a sequence $(d_i)_{i=0}^n$ of random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ of the form

$$(i)\ d_0 = f_0, \text{ and } (ii)\ d_i = f_i - f_{i-1} \text{ if } n \geqslant 1 \text{ and } i \in [n]$$

where $(f_i)_{i=0}^n$ is a martingale.

(P1) Monotone basic sequences in L_p for any $p \geqslant 1$, and orthogonal in L_2.

(P2) (Burkholder) Unconditional in L_p for any $p > 1$.

(P3) Satisfy a lower ℓ_2 estimate in L_p for any $1 < p \leqslant 2$, i.e.,

$$\Big(\sum_{i=0}^n \|d_i\|_{L_p}^2\Big)^{1/2} \leqslant \Big(\frac{4}{p-1}\Big) \cdot \Big\|\sum_{i=0}^n d_i\Big\|_{L_p}$$

8.b.1. Pseudorandomness: semirings

Definition

Let Ω be a nonempty set and k a positive integer. Also let $\mathcal{S}$ be a collection of subsets of Ω. We say that $\mathcal{S}$ is a k-**semiring** on Ω if the following properties are satisfied.

(a) We have that $\emptyset, \Omega \in \mathcal{S}$.

(b) For every $S, T \in \mathcal{S}$ we have that $S \cap T \in \mathcal{S}$.

(c) For every $S, T \in \mathcal{S}$ there exist $\ell \in [k]$ and pairwise disjoint sets $R_1, \dots, R_\ell \in \mathcal{S}$ such that $S \setminus T = R_1 \cup \cdots \cup R_\ell$.

If $f \in L_1(\Omega, \mathcal{F}, \mathbb{P})$ and $\mathcal{S} \subseteq \mathcal{F}$ is a k-semiring, then we set

$$\|f\|_{\mathcal{S}} = \sup\Big\{ \Big| \int_A f \, d\mathbb{P} \Big| : A \in \mathcal{S} \Big\}.$$

The quantity $\|f\|_{\mathcal{S}}$ will be called the $\mathcal{S}$-**uniformity norm** of f.

8.b.2. Pseudorandomness: semirings

- Every algebra of sets is a 1-semiring.
- The collection of all intervals of a totally ordered set is a 2-semiring.
- Let m be a positive integer and for every $i \in [m]$ let $\mathcal{S}_i$ be a k_i-semiring on Ω. Then the family

$$\mathcal{S} = \big\{X_1 \cap \cdots \cap X_m : X_i \in \mathcal{S}_i \text{ for every } i \in [m]\big\}$$

is a $\big(\sum_{i=1}^{m} k_i\big)$-semiring on Ω.

8.b.3. Pseudorandomness: semirings

Example

Let $d \geqslant 2$ and $V_1, \ldots, V_d$ nonempty finite sets.

The family

$$\mathcal{S}_{\min} = \{X_1 \times \cdots \times X_d : X_i \subseteq V_i \text{ for every } i \in [d]\}$$

of all rectangles of $V_1 \times \cdots \times V_d$ is a d-semiring.

The $\mathcal{S}_{\min}$-uniformity norm is known as the **cut norm** and was introduced by Frieze and Kannan. (Here, we view the product $V_1 \times \cdots \times V_d$ as a discrete probability space equipped with the uniform probability measure.)

8.b.4. Pseudorandomness: semirings

Example (cont'd)

Let $d \geqslant 2$ and $V_1, \dots, V_d$ nonempty finite sets.

For every $i \in [d]$ let $\mathcal{A}_i$ be the algebra of all subsets of $V_1 \times \cdots \times V_d$ not depending on the i-th coordinate. That is, $X \in \mathcal{A}_i$ if X is of the form $B \times V_i$ with $B \subseteq \prod_{j \neq i} V_j$.

Then the family

$$\mathcal{S}_{\max} = \big\{X_1 \cap \cdots \cap X_d : X_i \in \mathcal{A}_i \text{ for every } i \in [d]\big\}$$

is a d-semiring on $V_1 \times \cdots \times V_d$.

The $\mathcal{S}_{\max}$-uniformity norm is known as the **Gowers box norm** and was introduced by Gowers.

8.b.5. Pseudorandomness: semirings

Example

Let A be a finite alphabet with $|A| \geqslant 2$ and $n \geqslant 1$.

For every $\{a, b\} \in \binom{A}{2}$ let $\mathcal{A}_{\{a,b\}}$ be the algebra on A^n consisting of all subsets of A^n which are (a, b)-**insensitive** (Shelah).

Then the family $\mathcal{S}(A^n)$ defined by

$$X \in \mathcal{S}(A^n) \Leftrightarrow X = \bigcap_{\{a,b\} \in \binom{A}{2}} X_{\{a,b\}}$$

where $X_{\{a,b\}} \in \mathcal{A}_{\{a,b\}}$ for every $\{a, b\} \in \binom{A}{2}$, is a K-semiring on A^n with $K = |A|(|A| + 1)2^{-1}$.

8.c.1. Pseudorandomness: a decomposition of random variables

Theorem

Let k be a positive integer, $0 < \varepsilon \leqslant 1$ and $p > 1$. Also let $F\colon \mathbb{N} \to \mathbb{N}$ be an increasing function. Finally, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\mathcal{S}$ a k-semiring on Ω with $\mathcal{S} \subseteq \mathcal{F}$. Then for every $f \in L_p(\Omega, \mathcal{F}, \mathbb{P})$ with $\|f\|_{L_p} \leqslant 1$ there exist

(i) *a partition $\mathcal{P}$ of Ω with $\mathcal{P} \subseteq \mathcal{S}$ and $|\mathcal{P}| = O_{k,\varepsilon,p,F}(1)$, and*

(ii) *a decomposition $f = f_{\mathrm{str}} + f_{\mathrm{err}} + f_{\mathrm{unf}}$*

such that the following are satisfied.

(a) *The function f_{str} is constant on each $S \in \mathcal{P}$. Moreover, if f is non-negative, then both f_{str} and $f_{\mathrm{str}} + f_{\mathrm{err}}$ are non-negative.*

(b) *We have the estimates $\|f_{\mathrm{err}}\|_{L_p} \leqslant \varepsilon$ and $\|f_{\mathrm{unf}}\|_{\mathcal{S}} \leqslant \frac{1}{F(|\mathcal{P}|)}$.*

8.c.2. Pseudorandomness: a decomposition of random variables

The case "$p = 2$" is due to Tao (2006). His approach, however, is somewhat different since he works with σ-algebras instead of k-semirings. The general case is due to D, Kanellopoulos & Karageorgos (2014).

Applying this decomposition for various semirings we obtain:

- Szemerédi's regularity lemma – $\mathcal{S}_{\min}$;
- a regularity lemma for uniform hypergraphs – $\mathcal{S}_{\max}$;
- a regularity lemma for hypercubes – $\mathcal{S}(A^n)$;
- a regularity lemma for L_p graphons – $\mathcal{S}_{\min}$.

8.d.1. Pseudorandomness: a concentration inequality for product spaces

Let $(\Omega_1, \mathcal{F}_1, \mathbb{P}_1), \dots, (\Omega_n, \mathcal{F}_n, \mathbb{P}_n)$ be a finite sequence of probability spaces and denote by $(\boldsymbol{\Omega}, \boldsymbol{\mathcal{F}}, \boldsymbol{P})$ their product. More generally, for every nonempty $I \subseteq [n]$ by $(\boldsymbol{\Omega}_I, \boldsymbol{\mathcal{F}}_I, \boldsymbol{P}_I)$ we denote the product of the spaces $\{(\Omega_i, \mathcal{F}_i, \mathbb{P}_i) : i \in I\}$.

Let $I \subseteq [n]$ be such that I and $I^c := [n] \setminus I$ are nonempty. For every integrable random variable $f \colon \boldsymbol{\Omega} \to \mathbb{R}$ and every $\mathbf{x} \in \boldsymbol{\Omega}_I$ let $f_{\mathbf{x}} \colon \boldsymbol{\Omega}_{I^c} \to \mathbb{R}$ be the section of f at $\mathbf{x}$, that is, $f_{\mathbf{x}}(\mathbf{y}) = f(\mathbf{x}, \mathbf{y})$ for every $\mathbf{y} \in \boldsymbol{\Omega}_{I^c}$. Fubini's theorem asserts that the random variable $\mathbf{x} \mapsto \mathbb{E}(f_{\mathbf{x}})$ is integrable and satisfies

$$\int \mathbb{E}(f_{\mathbf{x}}) \, d\boldsymbol{P}_I = \mathbb{E}(f).$$

8.d.2. Pseudorandomness: a concentration inequality for product spaces

Theorem (D, Kanellopoulos, Tyros – 2014)

Let $0 < \varepsilon \leqslant 1$ *and* $1 < p \leqslant 2$*, and set*

$$c(\varepsilon, p) = \frac{\varepsilon^4 (p-1)^2}{38}.$$

Also let n *be a positive integer with* $n \geqslant c(\varepsilon, p)^{-1}$ *and let* $(\boldsymbol{\Omega}, \boldsymbol{\mathcal{F}}, \boldsymbol{P})$ *be the product of a finite sequence* $(\Omega_1, \mathcal{F}_1, \mathbb{P}_1), \dots, (\Omega_n, \mathcal{F}_n, \mathbb{P}_n)$ *of probability spaces. Then for every* $f \in L_p(\boldsymbol{\Omega}, \boldsymbol{\mathcal{F}}, \boldsymbol{P})$ *with* $\|f\|_{L_p} \leqslant 1$ *there exists an interval* $J \subseteq [n]$ *with* $J^{c} \neq \emptyset$ *and* $|J| \geqslant c(\varepsilon, p)n$*, such that for every nonempty* $I \subseteq J$ *we have*

$$\boldsymbol{P}_I\big(\{\mathbf{x} \in \boldsymbol{\Omega}_I : |\mathbb{E}(f_{\mathbf{x}}) - \mathbb{E}(f)| \leqslant \varepsilon\}\big) \geqslant 1 - \varepsilon.$$

8.d.3. Pseudorandomness: a concentration inequality for product spaces

Corollary

Let $0 < \varepsilon \leqslant 1$ *and* $1 < p \leqslant 2$. *If* $n \geqslant c(\varepsilon, p)^{-1}$ *and* $(\boldsymbol{\Omega}, \boldsymbol{\mathcal{F}}, \boldsymbol{P})$ *is the product of a finite sequence* $(\Omega_1, \mathcal{F}_1, \mathbb{P}_1), \dots, (\Omega_n, \mathcal{F}_n, \mathbb{P}_n)$ *of probability spaces, then for every* $A \in \boldsymbol{\mathcal{F}}$ *there exists an interval* $J \subseteq [n]$ *with* $J^{\mathrm{c}} \neq \emptyset$ *and* $|J| \geqslant c(\varepsilon, p)n$*, such that for every nonempty* $I \subseteq J$ *we have*

$$\boldsymbol{P}_I\Big(\big\{\mathbf{x} \in \boldsymbol{\Omega}_I : |\boldsymbol{P}_{I^{\mathrm{c}}}(A_{\mathbf{x}}) - \boldsymbol{P}(A)| \leqslant \varepsilon \boldsymbol{P}(A)^{1/p}\big\}\Big) \geqslant 1 - \varepsilon.$$

This result does not hold true for $p = 1$ (thus, the range of p in the previous theorem is optimal).

8.d.4. Pseudorandomness: a concentration inequality for product spaces

Applying this concentration inequality for various product spaces we obtain a (new type of) regularity lemma for a number of discrete structures such as:

- hypercubes;
- Carlson–Simpson spaces;
- polynomial Hales–Jewett spaces.

8.d.5. Pseudorandomness: a concentration inequality for product spaces

Lemma

Let k, m be positive integers with $k \geqslant 2$ and $0 < \varepsilon \leqslant 1$. Also let A be an alphabet with $|A| = k$ and n a positive integer with

$$n \geqslant \frac{76\, m\, k^{3m}}{\varepsilon^3}.$$

Then for every subset D of A^n there exists an interval $I \subseteq [n]$ with $|I| = m$ such that for ***every*** *$t \in A^I$ we have*

$$|\mathbb{P}_{A^{I^c}}(D_t) - \mathbb{P}(D)| \leqslant \varepsilon$$

where $D_t = \{s \in A^{I^c} : (t, s) \in D\}$ is the section of D at t.

(Here, all measures are uniform probability measures.)

9.a. Probabilistic versions (cont'd)

We can now give an outline of the method to obtain "local" probabilistic versions of density results (including, in particular, the density Hales–Jewett theorem).

STEP 1: By an application of Szemerédi's regularity method, we show that a given dense set D of our "structured" set $\mathcal{S}$ is sufficiently pseudorandom. This enables us to model the set D as a family of measurable events $\{D_t : t \in \mathcal{R}\}$ in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ indexed by a Ramsey space $\mathcal{R}$ closely related, of course, with $\mathcal{R}$. The measure of the events is controlled by the density of D.

9.b. Probabilistic versions (cont'd)

STEP 2: We apply coloring arguments and our basic density result to show that there exists a "substructure" $\mathcal{R}'$ of $\mathcal{R}$ such that the events in the subfamily $\{D_t : t \in \mathcal{R}'\}$ are highly correlated. The reasoning can be traced in an old paper of Erdős and Hajnal (1964).

STEP 3: We use a double counting argument to locate a "substructure" $\mathcal{S}'$ of $\mathcal{S}$ such that the set D contains a non-trivial portion of subsets of $\mathcal{S}'$ of the desired kind (combinatorial lines, Carlson–Simpson spaces, etc.).

9.c. Probabilistic versions (cont'd)

For every $k \geqslant 2$ and $0 < \delta \leqslant 1$ let $n_0 = \mathrm{DHJ}(k, \delta/2)$ and set

$$\zeta(k,\delta) = \frac{\delta/2}{(k+1)^{n_0} - k^{n_0}}.$$

Fact

If A is an alphabet with $|A| = k$, then for every combinatorial space W of $A^{<\mathbb{N}}$ of dimension at least n_0 and every family $\{D_w : w \in W\}$ of measurable events in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ satisfying $\mathbb{P}(D_w) \geqslant \delta$ for every $w \in W$, there exists a combinatorial line L of W such that

$$\mathbb{P}\Big(\bigcap_{w \in L} D_w\Big) \geqslant \zeta(k,\delta).$$

9.d. Probabilistic versions (cont'd)

Theorem (D, Kanellopoulos & Tyros – 2013)

For every pair k, m of positive integers with $k \geqslant 2$ and every $0 < \delta \leqslant 1$ there exists a positive integer $\mathrm{CorSp}(k, m, \delta)$ with the following property. If A is an alphabet with $|A| = k$, then for every combinatorial space W of $A^{<\mathbb{N}}$ with $\dim(W) \geqslant \mathrm{CorSp}(k, m, \delta)$ and every family $\{D_w : w \in W\}$ of measurable events in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ satisfying $\mathbb{P}(D_w) \geqslant \delta$ for every $w \in W$, there exists an m-dimensional combinatorial subspace V of W such that for every nonempty $F \subseteq V$ we have

$$\mathbb{P}\Big(\bigcap_{w \in F} D_w\Big) \geqslant \zeta(|F|, \delta).$$

Thanks for listening!