Construction of differentiable functions between Banach spaces.

joint work with P. Hajek, then with M. Ivanov and S. Lajara

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Relationship between the existence of non trivial real valued smooth functions on a separable Banach space X and the geometry of X.

Theorem. *Let X be a sepable Banach space. TFAE :*
(1) There exists on X an equivalent norm diff. on $X\setminus\{0\}$.
(2) There exists a $\mathcal{C}^1$-smooth function $b : X \to \mathbb{R}$ with bounded non empty support.
(3) X^* is separable.

Definition. X, Y Banach spaces. A function $f : X \to Y$ is G-differentiable at $x \in X$ if $\exists f'(x) \in \mathcal{L}(X, Y)$ such that for each $h \in X$, $\lim\limits_{t\to 0} \frac{f(x+th)-f(x)}{h} = f'(x)h$.

Theorem. *Let X be a sepable Banach space.*
(1) There exists on X an equivalent norm G-diff. on $X\setminus\{0\}$.
(2) There exists a G-diff. function $b : X \to \mathbb{R}$ with bounded non empty support.

Theorem [Azagra-Deville]. If X is an infinite dimensional Banach space with separable dual, there exists a $\mathcal{C}^1$-smooth real valued function on X with bounded support and such that $f'(X) = X^*$.

Theorem [Azagra,Deville and Jimenez-Sevilla]. Let X, Y be separable Banach spaces such that $dim(X) = \infty$. Then there exists $f : X \to Y$ Gâteaux-differentiable, such that $f'(X) = \mathcal{L}(X, Y)$.

Moreover, if $\mathcal{L}(X, Y)$ is separable, f can be chosen Fréchet-differentiable.

Theorem [Hajek]. If f is a function on c_0 with locally uniformly continuous derivative, then $f'(c_0)$ is included in a countable union of norm compact subsets of ℓ^1.

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Problem : Let X, Y be separable Banach spaces such that $dim(X) \geq 1$, $f : X \to Y$ differentiable at every point of X.
What is the structure of

$$f'(X) = \left\{ f'(x); x \in X \right\} \subset \mathcal{L}(X, Y)?$$

Is $f'(X)$ connected ?

Theorem : (Maly 96) : *If X is a Banach space and $f : X \to I\!R$ is Fréchet-differentiable at every point, then the set $f'(X)$ is connected in $(X^*, \|.\|)$.*

Let $f : I\!R^2 \to I\!R^2$, defined by :

$$f(x, y) = \left(x^2 \sqrt{y} \cos 1/x^3, x^2 \sqrt{y} \sin 1/x^3 \right)$$

if $(x, y) \neq (0, 0)$ and $f(0, 0) = (0, 0)$.
$\left\{ det(f'(x)); x \in I\!R^2 \right\} = \{0, 3/2\} \Rightarrow f'(I\!R^2)$ not connected.

Theorem : (T. Matrai) : *Let X be a separable Banach space, and let f be a real valued locally Lipschitz and Gâteaux-differentiable function on X. Then $f'(X)$ is connected in (X^*, w^*).*

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Theorem : (T. Matrai) : *Let X be a separable Banach space, and let f be a real valued locally Lipschitz and Gâteaux-differentiable function on X. Then $f'(X)$ is connected in (X^*, w^*).*

Proposition 1 : *If f is a continuous and Gâteaux-differentiable bump function on X, then the norm closure of $f'(X)$ contains a ball $B(r)$ for some $r > 0$.*

Proposition 2 :

Let X, Y be Banach spaces, $dim(X) \geq 1$.

Let $F : X \to Y$ be Lipschitz and Gâteaux-differentiable.

Assume that one of the following conditions hold :

(1) F is Lipschitz and $Y = \mathbb{R}$.

(2) Let F is Lipschitz and Fréchet-differentiable.

(3) $\mathcal{L}(X, Y)$ is separable.

Then, $\forall x \in X$, $\forall \varepsilon > 0$, $\exists y, z \in B_X(x, \varepsilon)$, $y \neq z$, such that

$$\|F'(y) - F'(z)\| \leq \varepsilon$$

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Proposition : *Let X be an infinite dimensional separable Banach space. Then, $\exists f : X \to \mathbb{R}$ Gâteaux-differentiable bump, such that f' is norm to weak* continuous and*

$$x \neq 0 \Rightarrow \|f'(0) - f'(x)\| \geq 1$$

If X^* is separable, we can assume moreover that f is $\mathcal{C}^1$ on $X \setminus \{0\}$.

Definition : Let X, Y be separable Banach spaces. (X, Y) has the jump property if $\exists F : X \to Y$ Lipschitz, everywhere G-differentiable, so that

$$\forall x, y \in X, x \neq y \Rightarrow \|F'(x) - F'(y)\| \geq 1$$

Question : When do (X, Y) possess the jump property ?

X, Y **separable Banach spaces.**

(1) $\mathcal{L}(X,Y)$ **is separable** $\Rightarrow$ (X,Y) **fails the jump property.**

(2) $(X,\mathbb{R})$ **fails the jump property.**

(3) $Y \subset Z$ and (X,Y) has the jump property $\Rightarrow$ (X,Z) has the jump property.

Theorem : $(\ell^1,\mathbb{R}^2)$ *has the jump property. More precisely,* $\exists F : \ell^1 \to I\!R^2$ *Gâteaux-differentiable, bounded, Lipschitz, such that for every* $x, y \in \ell^1$, $x \neq y$, *then*

$$\|F'(x) - F'(y)\|_{\mathcal{L}(\ell^1, I\!R^2)} \geq 1$$

Moreover, $\forall h \in \ell^1$, $x \to F'(x).h$ *is continuous from* ℓ^1 *into* $I\!R^2$.

Gâteaux-differentiability criterium : *Let X and Y be Banach spaces. Assume :*

** $f_n : X \to Y$ are G-differentiable.*
** $\left(\sum f_n\right)$ converges pointwise on X,*
** For all h, the series $\sum_{n \geq 1} \frac{\partial f_n}{\partial h}(x)$ converges uniformly in x.*

Then $f = \sum_{n \geq 1} f_n$ is G-differentiable on X, for all x, $f'(x) = \sum_{n \geq 1} f'_n(x)$ (where the convergence of the series is in $\mathcal{L}(X, Y)$ for the strong operator topology), and f is K-Lipschitz.

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Moreover, if each f'_n is continuous from X endowed with the norm topology into $\mathcal{L}(X, Y)$ with the strong operator topology, then f' shares the same continuity property.

Lemma : *Given* $p = (q, r) \in I\!R^2$ *such that* $q < r$ *and* $\varepsilon > 0$, *there exists a* $\mathcal{C}^\infty$*-function* $\varphi = \varphi_{p,\varepsilon} : I\!R^2 \to I\!R^2$ *such that :*

(i) $\quad |\varphi(x,y)| \leq \varepsilon \qquad$ *for all* $(x,y) \in I\!R^2$,

(ii) $\quad \varphi(x,y) = 0 \qquad$ *if* $x < q$,

(iii) $\quad \left\|\frac{\partial \varphi}{\partial x}(x,y)\right\| \leq \varepsilon \quad$ *for all* $(x,y) \in I\!R^2$,

(iv) $\quad \left\|\frac{\partial \varphi}{\partial y}(x,y)\right\| = 1 \quad$ *if* $x \geq r$,

(v) $\quad \left\|\frac{\partial \varphi}{\partial y}(x,y)\right\| \leq 1 \quad$ *for all* $(x,y) \in I\!R^2$,

Proof : $\varphi(x,y) = \frac{\beta(x)}{n}\Big(\sin(ny), \cos(ny)\Big)$,
with $\beta : \mathbb{R} \to [0,1]$ $\mathcal{C}^\infty$, $\beta(x) = 0$ if $x \leq q$ and $\beta(x) = 1$ if $x \geq r$.

Proof of Theorem : Let $\mathbb{P} = \{(q,r) \in \mathbb{Q}^2;\, q < r\}$ and $k \to (n_k, (q_k, r_k))$ be a bijection from $\mathbb{N}$ onto $\mathbb{N} \times \mathbb{P}$ such that for all k, $n_k \neq k$.

$\varepsilon > 0$, $\varepsilon_k > 0$ / $\sum\limits_{k=1}^{\infty} \varepsilon_k = \varepsilon$.

$f_k : \ell^1 \to I\!R^2$

$$f_k\Big(x\Big) = \varphi_{p_k, \varepsilon_k}\Big(x_{n_k}, x_k\Big)$$

f_k is a $\mathcal{C}^\infty$ function on ℓ^1.

$F : \ell^1 \to I\!R^2$ is defined by : $\quad F(x) = \sum_{k \in I\!N} f_k(x)$

- F is well-defined.

- F is G-differentiable on ℓ^1 and F is $(1+\varepsilon)$-Lipschitz on ℓ^1.

Indeed $\sum\limits_j \sup_{x \in \ell^1} \left\| \frac{\partial f_j}{\partial x_k} \right\| \leq \sum\limits_{j \neq k} \varepsilon_j + 1.$

- We claim that if $x \neq y \in \ell^1$, then $\|F'(x) - F'(y)\| \geq 1 - 2\varepsilon$.

$$f_k\Big(x\Big) = \varphi_{p_k,\varepsilon_k}\Big(x_{n_k}, x_k\Big)$$

- If $x \neq y \in \ell^1$, choose m such that (for example) $x_m \neq y_m$, then (q, r) such that $x_m < q < r < y_m$ and finally k such that $(n_k, q_k, r_k) = (m, q, r)$.

$$\frac{\partial f_k}{\partial x_k}(x) = 0 \qquad \Big\|\frac{\partial f_k}{\partial x_k}(y)\Big\| = 1$$

and, if $j \neq k$,

$$\frac{\partial f_j}{\partial x_k}(x) \leq \varepsilon_j \qquad \Big\|\frac{\partial f_j}{\partial x_k}(y)\Big\| \leq \varepsilon_j$$

Therefore

$$\|F'(x) - F'(y)\|_{\mathcal{L}(\ell^1, I\!R^2)} \quad \geq \quad \|\frac{\partial F}{\partial x_k}(x) - \frac{\partial F}{\partial x_k}(y)\|_{I\!R^2}$$

$$\geq \quad \|\frac{\partial f_k}{\partial x_k}(x) - \frac{\partial f_k}{\partial x_k}(y)\| - \sum_{j \neq k} \|\frac{\partial f_j}{\partial x_k}(x) - \frac{\partial f_j}{\partial x_k}(y)\| \quad \geq \quad 1 - 2\varepsilon$$

Theorem. *Let X, Y be separable Banach spaces. Assume :*
$(e_n, e_n^*) \subset X \times X^*$ is a total, bounded, biorthogonal system,
$(f_n) \subset Y$ is an unconditional basic sequence such that :
$\forall h \in X$, $\left(\sum e_n^*(h) f_{2n-1}\right)$ and $\left(\sum e_n^*(h) f_{2n}\right)$ converge in norm.
Then (X, Y) has the jump property.

Proof. Define $z_k : X \to \mathbb{R}^2$ by $z_k(x) = \left(e_{n_k}^*(x), e_k^*(x)\right)$ then
$i_k : \mathbb{R}^2 \to Y$ by $i_k(s,t) = t f_{2k-1} + s f_{2k}$,
$F_k : X \to Y$ by $F_k = i_k \circ \varphi_{p_k, \varepsilon_k} \circ z_k$ and $F = \sum F_k$.

Corollary (Bayart). *If X is a separable, infinite dimensional Banach space, then (X, c_0) has the jump property.*

Corollary. *Let* X *be a Banach space with a Schauder basis* (e_n)*,* Y *be a Banach space and* $U \in \mathcal{L}(X,Y)$ *such that* $\big(U(e_n)\big)$ *is a subsymmetric basis. Then* (X,Y) *has the jump property.*

Example. *Let* $X_p = \ell^p$ *if* $1 \leq p < +\infty$ *and* $X_\infty = c_0$*. Let us fix* $1 \leq p, q \leq +\infty$*. TFAE :*

(1) (X_p, X_q) *has the jump property.*

(2) $p \leq q$*.*

(3) $\mathcal{L}(X_p, X_q)$ *is not separable.*

Example. *Let* J *be the James' space. Then* (J, ℓ^2) *and* (J, J) *have the jump property.*

Corollary. *Let* X *be a Banach space with a Schauder basis* (e_n), Y *be a Banach space such that* $Y \approx Y \oplus Y$ *and* $U \in \mathcal{L}(X, Y)$ *such that* $\big(U(e_n)\big)$ *is an unconditional basis. Then* (X, Y) *has the jump property.*

Example. *Assume* $1 \leq q \leq p \leq 2$ *and* $p \neq 1$. *Then* $\big(L^p([0,1]), L^q([0,1])\big)$ *has the jump property.*

What about the other values of p *and* q *?*

Corollary. *Let* X *be a Banach space with an unconditional basis and such that* $X \approx X \oplus X$. *Then* (X, X) *has the jump property.*

Example. *If* T *is the Tsirelson space, then* (T, T) *and* (T^*, T^*) *have the jump property. If* X *is the space of Argyros and Haydon, then* (X, X) *fails the jump property.*

Open questions

1) *Does* $(L^1([0,1]), L^1([0,1])$ *have the jump property ?*

2) *If* $\mathcal{L}(X,Y)$ *is nonseparable and* $dim(Y) \geq 2$*, does* (X,Y) *have the jump property ?*

If $\mathcal{L}(X,Y)$ *icontains* ℓ^∞ *and* $dim(Y) \geq 2$*, does* (X,Y) *have the jump property ?*

3) *Does* $(JT, \mathbb{R}^2)$ *have the jump property ?*

4) *Describe the couples* (X,Y) *of separable Banach spaces for which*

$\exists (e_n, e_n^*) \subset X \times X^*$ *is a total, bounded, biorthogonal system,*
$\exists (f_n) \subset Y$ *is an unconditional basic sequence such that :*
$\forall h \in X$, $\left(\sum e_n^*(h) f_n\right)$ *converges in norm.*
(this imply $\mathcal{L}(X,Y) \supset \ell^\infty$*)*

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