Compatible complex structures on Kalton-Peck space

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August 25, 2014
First Brazilian Workshop in Geometry of Banach Spaces

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These spaces admit at least two complex structures.

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Every twisted sum is equivalent to one of the type $X \oplus_\Omega Y$ for a quasi-linear operator $\Omega : Y \to X$.

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$Z_2 = \ell_2 \oplus_{\Omega_2} \ell_2$ is the twisted Hilbert space obtained by considering the non-trivial quasi-linear map (defined on finitely supported sequences)

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Question

Z_2 is isomorphic to its hyperplanes?

Complex structures on Z_2

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- $Z_2^{(u_2,u_2)}$, *where* $(u_2, u_2)(x, y) = (u_2 x, u_2 y)$.

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Question

Does Z_2 *admit unique complex structure?*

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An essential element to prove this is the following result:

Theorem (V. Ferenczi, E. Galego, 2007)

Let T, u be complex structures on, respectively, an infinite dimensional Banach space X and some hyperplane H of X. Then the operator $T|_H - u$ is not strictly singular.

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Definition

The pair of operators (α, β)

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The pair of operators (α, β) is said to be compatible with Ω if there exists an operator γ such that the diagram is commutative.

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Example. The pair (u_2, u_2) is compatible with Ω_2.

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For every operator $T : \ell_2 \to \ell_2$, and for every block subspace W of ℓ_2, the commutator $\Omega_2 T - T\Omega_2$ is trivial on some block subspace of W.

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Sketch of the proof: Let u be a complex structure on ℓ_2. Suppose that can be extended to U on $\ell_2 \oplus_{\Omega_2 i} H$

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Extending v to a complex structure V on ℓ_2, we have that (u, V) is compatible with Ω_2. Then $u - V$ is compact.

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Proposition

Let (T, U) be a pair of compatible operators on Z_2. Then $T - U$ is compact.

Sketch of the proof: Let u be a complex structure on ℓ_2. Suppose that can be extended to U on $\ell_2 \oplus_{\Omega_2 i} H$

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \ell_2 & \longrightarrow & \ell_2 \oplus_{\Omega_2 i} H & \longrightarrow & H & \longrightarrow & 0 \\
 & & \downarrow u & & \downarrow U & & \downarrow v & & \\
0 & \longrightarrow & \ell_2 & \longrightarrow & \ell_2 \oplus_{\Omega_2 i} H & \longrightarrow & H & \longrightarrow & 0
\end{array}$$

Extending v to a complex structure V on ℓ_2, we have that (u, V) is compatible with Ω_2. Then $u - V$ is compact.
On the other side, it follows from Ferenczi-Galego theorem that $u|_H - v$ is not strictly singular. So we get a contradiction.

References

J. Bourgain.
Real isomorphic complex Banach spaces need not be complex isomorphic
Proc. Amer. Math. Soc. 96 **(2)** (1986), 221–226.

F. Cabello Sánchez, J.M.F. Castillo, J. Suárez.
On strictly singular nonlinear centralizers
Nonlinear Anal. 75 (2012), 3313–3321.

V. Ferenczi, E. Galego.
Even infinite-dimensional real Banach spaces
J. Funct. Anal 253 **(2)** (2007), 534–549.

N. J. Kalton.
An elementary example of a Banach space not isomorphic to its complex conjugate
Canad. Math. Bull. **(38)** (1995), 218–222.

References

N.J. Kalton, N.T. Peck.
Twisted sums of sequence spaces and the three space problem
Trans. Amer. Math. Soc. 255 (1979) 1–30.

J. Lindenstrauss and L. Tzafriri
Classical Banach spaces I, sequence spaces
Ergeb. Math. 92, Springer-Verlag (1977).

S. Szarek.
On the existence and uniqueness of complex structure and spaces with 'few' operators
Trans. Amer. Math. Soc. 293 **(1)** (1986), 339–353.