Compatible complex structures on Kalton-Peck space

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Let X be a real Banach space such that $X \oplus_{\mathbb{C}} X$ is primary, then X has at most one complex structure.

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Every twisted sum is equivalent to one of the type $X \oplus_{\Omega} Y$ for a quasi-linear operator $\Omega: Y \to X$.

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 $Z_2 = \ell_2 \oplus_{\Omega_2} \ell_2$ is the twisted Hilbert space obtained by considering the non-trivial quasi-linear map (defined on finitely supported sequences)

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Properties

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Question

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- $Z_2(\mathbb{C}) = \ell_2(\mathbb{C}) \oplus_{\Omega_2^{\mathbb{C}}} \ell_2(\mathbb{C}).$

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Question

Does Z_2 admit unique complex structure?

Theorem

No complex structure on ℓ_2 can be extended to a complex structure on the hyperplane $\ell_2 \oplus_{\Omega_2 i} H$.

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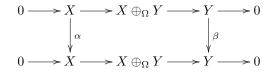
There exists a complex structure U on ℓ_2 that can not be extended to any operator on Z_2 .

An essential element to prove this is the following result:

Theorem (V. Ferenczi, E. Galego, 2007)

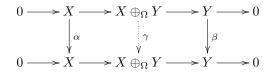
Let T, u be complex structures on, respectively, an infinite dimensional Banach space X and some hyperplane H of X. Then the operator $T|_H - u$ is not strictly singular.

Definition The pair of operators (α, β)



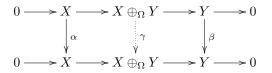
Definition

The pair of operators (α, β) is said to be compatible with Ω if there exists an operator γ such that the diagram is commutative.



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Example. The pair (u_2, u_2) is compatible with Ω_2 .

Proposition

For every operator $T: \ell_2 \to \ell_2$, and for every block subspace W of ℓ_2 , the commutator $\Omega_2 T - T\Omega_2$ is trivial on some block subspace of W.

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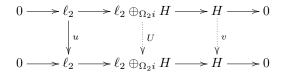
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