Tsirelson spaces with constraints

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Schreier Families

Let

$$\mathcal{S}_1 = \{F \subset \mathbb{N} : |F| \leqslant \min F\}.$$

If α is a countable ordinal and $\mathcal{S}_\alpha$ has been defined let

$$\mathcal{S}_{\alpha+1} = \{\bigcup_{i=1}^{n} F_i : F_1 < \cdots < F_n, (\min F_i)_{i=1}^{n} \in \mathcal{S}_1, F_i \in \mathcal{S}_\alpha\}.$$

If ξ is a limit ordinal let $\xi_n \uparrow \xi$ and define

$$\mathcal{S}_\xi = \{F : \exists n \leqslant F \in \mathcal{S}_{\xi_n}\}.$$

Spreading Models

A sequence (x_n) in a Banach space generates an $\mathcal{S}_\alpha$-ℓ_p spreading model if there is a constant $C > 0$ so that for all scalars $(a_i)_i$ and $F \in \mathcal{S}_\alpha$ we have

$$\frac{1}{C}\Big(\sum_{i\in F}|a_i|^p\Big)^{1/p} \leqslant \Big\|\sum_{i\in F}a_i x_i\Big\| \leqslant C\Big(\sum_{i\in F}|a_i|^p\Big)^{1/p}.$$

Strictly Singular Operators

A operator $T \in \mathcal{L}(X)$ is $\mathcal{S}_\alpha$-strictly singular ($T \in \mathcal{SS}_\alpha(X)$) if for every $\varepsilon > 0$ and normalized basic sequence $(x_n)_{n\in\mathbb{N}}$ in X

$$\exists F \in \mathcal{S}_\alpha \text{ and } x \in S_{[x_n]_{n\in F}} \text{ satisfying } \|Tx\| < \varepsilon.$$

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For each α the set $\mathcal{SS}_\alpha(X)$ is norm closed,

$$\mathcal{K}(X) \subset \mathcal{SS}_\alpha(X) \subset \mathcal{SS}_\beta(X) \subset \mathcal{SS}(X)$$

for $\alpha < \beta$. For $T \in \mathcal{L}(X)$ and $S \in \mathcal{SS}_\alpha(X)$ we have $TS, ST \in \mathcal{SS}_\alpha(X)$.
For separable X we have

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However, it is not necessarily true that $S_1, S_2 \in \mathcal{SS}_\alpha(X)$ implies that $S_1 + S_2 \in \mathcal{SS}_\alpha(X)$ [Odell and Teixeira].

Distortion

A Banach space $(X, \|\cdot\|)$ with a basis $[e_n]$ is $\mathcal{S}_\alpha$-*arbitrarily distortable* if for every $\lambda > 0$ there is an equivalent norm $|\cdot|$ on X so that for every block sequence (x_n) of (e_n)

$$\exists F \in \mathcal{S}_\alpha \text{ and } x, y \in S_{[x_n]_{n\in F}} \text{ with } \frac{|x|}{|y|} > \lambda.$$

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Define

$$AD(X) = \min\{\alpha : X \text{ is } \mathcal{S}_\alpha\text{-arbitraily distortable}\}$$

We have $AD(X) < \omega_1$ if and only if X is arbitrarily distortable.

Tsirelson space

Let $0 < \theta < 1$. The space T is the completion of c_{00} under a norm that satifies the following implicit equation: Let $x \in c_{00}$

$$\|x\| = \max\{\|x\|_\infty, \sup \theta \sum_{i=1}^{n} \|E_i x\|\}$$

where the supremum is for $n \in \mathbb{N}$ and successive intervals $(E_i)_{i=1}^n$ satisfying

$$(\min E_i)_{i=1}^n \in \mathcal{S}_1.$$

A Tsirelson space with Constraints

Let $0 < \theta < 1$. The space $T_{0,1}$ is the completion of c_{00} with respect to the norm satisfying the following implicit equation. Let $x \in c_{00}$

$$\|x\| = \max\{\|x\|_\infty, \sup \theta \sum_{i=1}^{n} \|E_i x\|_{m_i}\}$$

where the supremum is for successive intervals $(E_i)_{i=1}^n$ and $(m_i)_{i=1}^n$ with

$$(\min E_i)_{i=1}^n \in \mathcal{S}_1 \text{ and } m_i > \max E_{i-1}$$

where

$$\|x\|_m = \sup \frac{\theta}{m} \sum_{j=1}^{m} \|F_j x\|.$$

where the supremum is with respect to any successive intervals $(F_j)_{j=1}^m$.

A Tsirelson space with Constraints

Let $0 < \theta < 1$. The space $T^{k}_{0,1}$ is the completion of c_{00} with respect to the norm satisfying the following implicit equation. Let $x \in c_{00}$

$$\|x\| = \max\{\|x\|_\infty, \sup \theta \sum_{i=1}^{n} \|E_i x\|_{m_i}\}$$

where the supremum is for successive intervals $(E_i)_{i=1}^n$ and $(m_i)_{i=1}^n$ with

$$(\min E_i)_{i=1}^n \in \mathcal{S}_k \text{ and } m_i > \max E_{i-1}$$

where

$$\|x\|_m = \sup \frac{\theta}{m} \sum_{j=1}^{m} \|F_j x\|.$$

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A Tsirelson space with Constraints

Let $0 < \theta < 1$. The space $T_{0,1}^k$ is the completion of c_{00} with respect to the norm satisfying the following implicit equation. Let $x \in c_{00}$

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A Tsirelson space with Constraints

Let $0 < \theta < 1$ and α be a countable ordinal. The space $T_{0,1}^{\omega^\alpha}$ is the completion of c_{00} with respect to the norm satisfying the following implicit equation. Let $x \in c_{00}$

$$\|x\| = \max\{\|x\|_\infty, \sup \theta \sum_{i=1}^{n} \|E_i x\|_{m_i}\}$$

where the supremum is for successive intervals $(E_i)_{i=1}^n$ and $(m_i)_{i=1}^n$ with

$$(\min E_i)_{i=1}^n \in \mathcal{S}_{\omega^\alpha} \text{ and } m_i > \max E_{i-1}$$

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where

$$\|x\|_m = \sup \frac{\theta}{m} \sum_{j=1}^{m} \|F_j x\|.$$

where the supremum is with respect to any successive intervals $(F_j)_{j=1}^m$.

A Tsirelson space with Constraints

Let $0 < \theta < 1$ and $1 \leqslant p < q \leqslant \infty$. The space $T_{p,q}$ is the completion of c_{00} with respect to the norm satisfying the following implicit equation. Let $x \in c_{00}$

$$\|x\| = \max\{\|x\|_\infty, \sup \theta \Big(\sum_{i=1}^{n} \|E_i x\|_{m_i,q}^p \Big)^{\frac{1}{p}}\}$$

where the supremum is for successive intervals $(E_i)_{i=1}^n$ and $(m_i)_{i=1}^n$ with

$$(\min E_i)_{i=1}^n \in \mathcal{S}_1 \text{ and } m_i > \max E_{i-1}$$

where

$$\|x\|_{m,q} = \sup \frac{\theta}{m^{1/q'}} \sum_{j=1}^{m} \|F_j x\|.$$

where the supremum is with respect to any successive intervals $(F_j)_{j=1}^m$.

A Tsirelson space with Constraints

Let $0 < \theta \leqslant \frac{1}{2}$ and $1 \leqslant p < q \leqslant \infty$. The space $T_{p,q}$ is the completion of c_{00} with respect to the norm satisfying the following implicit equation. Let $x \in c_{00}$

$$\|x\| = \max\{\|x\|_\infty, \sup \theta \Big(\sum_{i=1}^{n} \|E_i x\|_{m_i,q}^p \Big)^{\frac{1}{p}}, \sup \theta \Big(\sum_{i=1}^{n} \|F_i x\|^q \Big)^{\frac{1}{q}} \}$$

where the supremums are for successive intervals $(E_i)_{i=1}^n$ and $(m_i)_{i=1}^n$ with

$$(\min E_i)_{i=1}^n \in \mathcal{S}_1 \text{ and } m_i > \max E_{i-1}$$

and any successive intervals $(F_i)_{i=1}^n$. Also

$$\|x\|_{m,q} = \sup \frac{\theta}{m^{1/q'}} \sum_{j=1}^{m} \|F_j x\|$$

where the supremum is for any successive intervals $(F_j)_{j=1}^m$.

Additional Constructions

- Odell and Schlumprecht constructed a space so that every monotone basic sequence is block finitely represented in every subspace.
- Argyros and Motakis constructed an HI space so that in every subspace every subsymmetric basic sequence is admitted as a spreading model (with constant 148, of course)
- Argyros and Motakis constructed a reflexive HI space so that any operator on any infinite dimensional subspace has a non-trivial invariant subspace.

Open Problems

1. Does there exist a space X so that $AD(X) = 1$? Such a space cannot contain an ℓ_1 or c_0 spreading model.
2. Does there exist an arbitrarily distortable space X which does not admit a c_0 or ℓ_1 spreading model satisfying $AD(X) > 1$.
3. Is $AD(\ell_2) > 1$? (Gowers)
4. Which closed subsets of $[1, \infty]$ can be hereditary Krivine p sets?
5. For $1 < p < q < \infty$ is the space $T_{p,q}$ superreflexive?

Thank you for listening!