# Generalizations of Gowers' Theorem

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BWB 2014
Maresias
August 25-29, 2014

This work was supported by the grant FAPESP 2013/14458-9.

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$\mathrm{P}S(c_0)$ - positive part of the sphere of $c_0$

# Discretization

$$\{f:\mathbb{N}\longrightarrow\{1,(1+\varepsilon)^{-1},\dots,(1+\varepsilon)^{-(k-1)}\},|\mathrm{supp}(f)|<\aleph_0,\\ \exists n\in\mathbb{N}\ f(n)=1\}=:\mathrm{FIN}_k$$

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$f \in \mathrm{FIN}_k \rightsquigarrow (c(f) = i \longleftrightarrow F(f) \in I_i)$

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Partial addition

$\mathrm{supp}(p) \cap \mathrm{supp}(q) = \emptyset \ \longrightarrow \ p + q(n) = \max\{p(n), q(n)\}$

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**Theorem (Hindman)**

*Let $c : \mathrm{FIN}(\mathbb{N}) \longrightarrow \{1, 2, \ldots, r\}$ be a finite colouring. Then there is an infinite $A \subset \mathrm{FIN}(\mathbb{N})$ such that $\mathrm{FU(A)}$ is monochromatic.*

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for some $l \in \mathbb{N}$, $b_s \in B$, $j_s \in \{0, 1, \ldots, k\}$, and at least one $j_s = 0$.

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*For every $k, m, r$, there exists $n$ such that for every colouring $c : \mathrm{FIN}_k(n) \longrightarrow \{1, 2, \ldots, r\}$ there is a block sequence $B \subset \mathrm{FIN}_k(n)$ of length $m$ such that $\langle B \rangle$ is monochromatic.*

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**Theorem (Ojeda-Aristizabal)**

*$g_k(m, 2) \leq f_{4+2(k-1)} \circ f_4(6m-2)$, where $f_i$ is the $i$-th function in the Ackermann Hierarchy.*

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- types of $T_1(p)$ and $T_2(p)$ are the same
- We can find a monochromatic subsequence in $\langle T_1(C) \rangle$ .

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**Question**

Does $\mathcal{F}_<$ satisfy the Ramsey property?

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**Theorem**

*$\mathcal{F}_<$ satisfies the Ramsey property, i.e., for every $A, B \in \mathcal{F}_<$ there exists $C \in \mathcal{F}_<$ such that for every colouring*

$$c : \{C \longrightarrow A\} \longrightarrow \{1, 2, \ldots, r\}$$

*there exists $f : C \longrightarrow B$ such that $\{B \longrightarrow A\} \circ f$ is monochromatic.*

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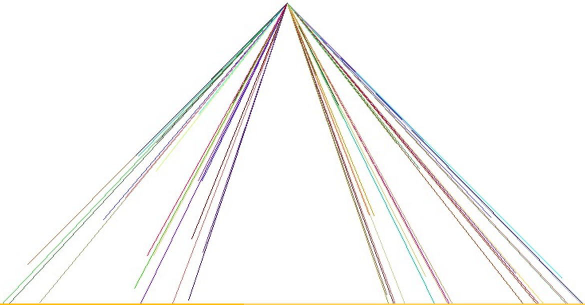

# Translation to $\mathrm{FIN}_k$

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$\phi : C \longrightarrow A \rightsquigarrow p_\phi(i) = \max(\phi(b_i)) \in \mathrm{FIN}_k(n)$

# Translation to $\mathrm{FIN}_k$

$A$ - path of length $k$ ($\equiv \{0, 1, \ldots, k\}$ - 0 the root)
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$\phi : C \longrightarrow A \rightsquigarrow p_\phi(i) = \max(\phi(b_i)) \in \mathrm{FIN}_k(n)$

## $\mathrm{FIN}_{\mathrm{k,l}}$

*Let $k, m, r$ and $l \geq k$ be natural numbers. Then there exists a natural number $n$ such that whenever we have a colouring $c : \mathrm{FIN}_k(n) \longrightarrow \{1, 2, \ldots, r\}$, there is a block sequence $B$ in $\mathrm{FIN}_l(n)$ of length $m$ such that the partial semigroup*

$$\left\langle \bigcup_{\vec{i} \in P^l_{k+1}} T_{\vec{i}}(B) \right\rangle$$

*is monochromatic.*

# Higher dimensions

$\mathrm{FIN}_k^{[d]}(n)$ = block sequences in $\mathrm{FIN}_k(n)$ of length $d$

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$\mathrm{FIN}_k^{[d]}(n)$ = block sequences in $\mathrm{FIN}_k(n)$ of length $d$

**Theorem**

*Let $(d, k, m, r)$ be a tuple of natural numbers. There exists $n$ such that for every colouring $c : \mathrm{FIN}_k^{[d]}(n) \longrightarrow \{0, 1, \ldots, r\}$, there is a block sequence $B$ of length $m$ such that $\langle B \rangle^{[d]}$ is monochromatic.*

### Question

Do our results admit infinitary versions?

**Question**

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Are there other applications of the new operations (.... to Banach spaces)?

# The end

THANK YOU FOR YOUR ATTENTION!