On Separable Quotients

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OPEN PROBLEM:(SQP)

The Problem

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Given an infinite Banach space E, show that there exists a closed subspace X so that E/X is isomorphic

To an infinite-dimensional separable Banach space.

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- Bessaga-Pelczynski (1958): Spaces whose dual contain c_0.
- Pelczynski (1964): Reflexive spaces.
- Johnson-Rosenthal (1972): Separable Spaces.
- Hagler-Johnson (1977): Spaces whose dual have unconditional basic sequences.
- Plichko (1980): Spaces with fundamental biorthogonal systems.
- Argyros-Dodos-Kanellopoulos (2008): Dual spaces.

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Remark: For many other results including characterizations, variants and recent progresses, see Mujica's survey and recent work of Argyros-Dodos-Kanellopoulos and Dodos, Lopez-Abad and Todorcevic.

Results From "*MathOverFlow*" - *BJ*

- *If* X^* *has no HI subspace, then* X *has a separable quotient.*
- *If* X^* *is weak***-separable, then* X *has a separable quotient.*

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Open Question (Godunov, 1974)

Let E be a Banach space. Then there exists a continuous vector field $f : \mathbb{R} \times E \to E$ so that

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Positive Answers:

- Godunov (1974) for the Hilbert space $E = \ell_2$.
- Shkarin (2003) solved for Banach space having complemented subpsaces with unconditional Schauder basis.
- Hájek-Johanis (Best Answer (2010)): For spaces having Nontrivial Separable Quotients.

Weak*-Transfinite Schauder Frame

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- $\forall\ y^* \in \overline{\text{span}}^{w^*}\Big\{f_\alpha : \alpha < \xi\Big\}\ \ \exists\ (a_\alpha(y^*))_{\alpha<\xi} \in \ell_\infty(\xi)$ s.t.

$$\langle y^*, x\rangle = \lim_{\alpha\to\xi}\langle\sum_{\gamma=0}^{\alpha} a_\gamma(y^*) f_\gamma, x\rangle$$

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$$\langle y^*, x\rangle = \lim_{\alpha\to\xi} \langle \sum_{\gamma=0}^{\alpha} a_\gamma(y^*) f_\gamma, x\rangle$$

- $(f_\alpha)_{\alpha<\xi}$ admits a biorthogonal system $(e_\alpha)_{\alpha<\xi}$ in E.

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$$Z = \Big\{x \in E : \ \sum_{\alpha<\xi} |f_\alpha(x)| \|e_\alpha\| < \infty\Big\} \text{ is dense in } E$$

ℓ_1-FUNDAMENTAL SYSTEMS

A biorthogonal system $\{x_\alpha; x_\alpha^*\}_{\alpha\in\Gamma}$ in $E \times E^*$ is called ℓ_1-fundamental if the linear space

$$\left\{x \in E : \sum_{\alpha\in\Gamma} |x_\alpha^*(x)|\|x_\alpha\| < \infty\right\}$$

is dense in E.

Remark. Every Fundundamental Biorthogonal System if ℓ_1-fundamental.

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- (I) $U \subset X$ is called a barrel in X is it is closed, absolutely convex and absorbing.
- (II) X is called *barreled* if every barrel in X is a neighborhood of 0.
- (III) **Closed Graph Theorem**: If X is barreled, Y is Fréchet and $T\colon X \to Y$ is a closed linear map, then T is continuous.
- (IV) **Known Characterizaiton**: A Banach space X has an infinite-dimensional separable quotient **iff** X has a non-barreled proper dense subspace.

Theorem

Every Banach space E with a ℓ_1-fundamental biorthogonal system has a non-trivial separable quotient.

Suppose that this is not so. Then E does not contain ℓ_1 and, moreover, as the linear space

Continuation of the "Proof"

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it is barreled. Define now the linear operator

$$T \colon Z \to \ell_1(\xi); \quad T(x) = \big(f_\alpha(x) \|e_\alpha\| \big)_{\alpha<\xi}, \quad x \in E.$$

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$$T : Z \to \ell_1(\xi); \quad T(x) = \big(f_\alpha(x) \|e_\alpha\| \big)_{\alpha < \xi}, \quad x \in E.$$

One readily shows that T has closed linear graph. Since Z is barreled, T is continuous.

- As T is bounded, it can be linearly extended to the whole space E.
- Denote this extension by T, too.
- Since T is not compact, $T(B_E)$ contains a semi normalized sequence (x_n) which is equivalent to the unit basis of ℓ_1.
- By the lifting property, the formal inverse T^{-1} from $\text{span}\{x_n\}$ back to E is bounded.
- Thus, T^{-1} is really the inverse of T.
- $\{T^{-1}(x_n)\}$ has a subsequence equivalent to the unit basis of ℓ_1.
- Contradiction.

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Thanks!