# A classification of separable Banach spaces under analytic determinacy

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Universidad de Murcia, Author supported by MEyC and FEDER under project MTM2011- 25377

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Proof: Theorem 1 + Ramsey (Louveau, Milliken, A.-Todorcevic...)

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**Axiom of Analytic Determinacy ($\mathbf{\Sigma^1_1 D}$)**

If $A$ is either analytic or coanalytic, then one player has a winning strategy in $G_A$.

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Let $\mathscr{K}(B_X) = \{L \subset B_X, L \text{ weakly compact}\}$ ordered by $\subset$

**Theorem 1 (APR)**

($\mathbf{\Sigma^1_1 D}$) If $X$ is separable Banach space, either

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Let $\mathscr{R}(A) = \{L \subset A, L \text{ relatively weakly compact}\}$ ordered by $\subset$

**Theorem 1a**

($\mathbf{\Sigma^1_1 D}$) If $X$ is Banach, and $A \subset X$ is countable, either

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In other models of set theory, there exists an unconditional basis $A$ not fitting in the list.

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In other models of set theory, there exists an unconditional basis $A$ not fitting in the list. More precisely, if there exists a coanalytic set of size $\omega_1$, then there exists $A$ with $\mathscr{R}(A) \sim Fin(\omega_1)$.

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# Combinatorial result behind Theorem 1

$I^{\perp} = \{a \subset \mathbb{N} : \forall b \in I \ a \cap b \text{ is finite}\}$

**Theorem**

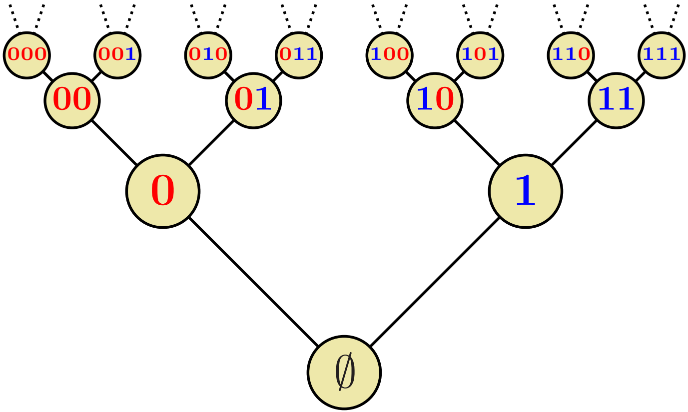
($\mathbf{\Sigma^1_1 D}$) If $I$ is an analytic family of subsets of $\mathbb{N}$, then $I^{\perp}$ is Tukey equivalent to either $\{0\}$, $\mathbb{N}$, $\mathbb{N}^{\mathbb{N}}$, $\mathscr{K}(\mathbb{Q})$ or $Fin(\mathbb{R})$.

Proof:

1. By a modification of a result of Todorcevic, either we get a special copy of the dyadic tree (gives $Fin(\mathbb{R})$) or $I$ and $I^{\perp}$ are countably separated.
2. By results of A. and Todorcevic, if $I$ and $I^{\perp}$ are countably separated, we can identify $I^{\perp}$ with $\mathscr{K}(E)$ and then apply Fremlin's theorem.

# The dyadic tree

The dyadic tree $2^{<\omega}$ is the set of finite sequences of 0's and 1's.



# The dyadic tree

A 0-chain is a subset $\{x_1, x_2, \ldots\} \subset 2^{<\omega}$ in which $x_{n+1} = x_n {}^\frown 0 {}^\frown y_n$

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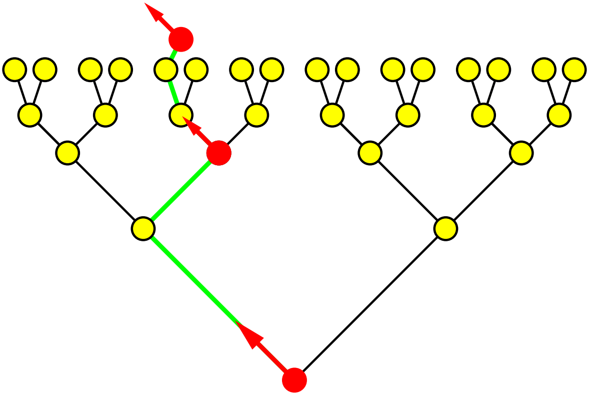

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A 1-chain is a subset $\{x_1, x_2, \dots\} \subset 2^{<\omega}$ in which $x_{n+1} = x_n{}^\frown 1^\frown y_n$

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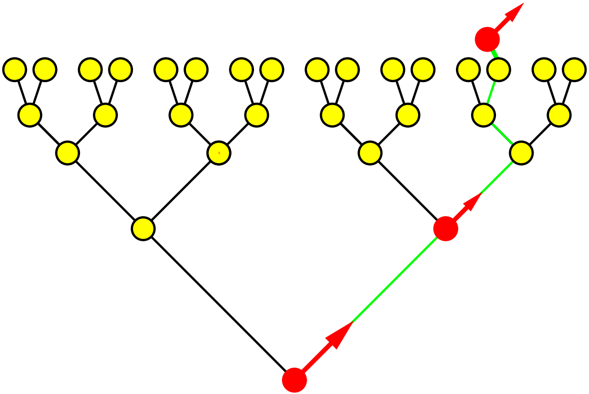

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- Condition (2) leads to $I^\perp \sim I \sim Fin(2^\omega)$.

# Tukey classification of $\mathscr{AK}(B_X)$

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- Let $\mathscr{AK}(B_X) = \{L \subset B_X, L \text{ weakly compact}\}$ endowed with multiple relations $L \leq_\varepsilon L'$ if $L \subset L' + \varepsilon B_X$.
- $\mathscr{AK}(B_X) \preceq \mathscr{AK}(B_Y)$ now means that there exist functions $f_\varepsilon : \mathscr{AK}(B_X) \longrightarrow \mathscr{AK}(B_Y)$ such that

  $\forall \varepsilon \quad \exists \delta \quad f_\varepsilon : (\mathscr{AK}(B_X), \leq_\varepsilon) \longrightarrow (\mathscr{AK}(B_Y), \leq_\delta)$ is Tukey.

**Theorem 2 (APR)**

Under the axiom of analytic determinacy, either

1. $\mathscr{AK}(B_X) \sim \{0\}$,
2. $\mathscr{AK}(B_X) \sim \mathbb{N}$,
3. $\mathscr{AK}(B_X) \sim \mathbb{N}^\mathbb{N}$,
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Proof: Theorem 1 + Ramsey (Louveau, Milliken, A.-Todorcevic...)

# Illustration of the use of Ramsey

- At a stage, we have $\mathscr{K}(\mathbb{Q}) \preceq \mathscr{K}(B_X)$ and we want to prove $\mathscr{K}(\mathbb{Q}) \preceq \mathscr{A}\mathscr{K}(B_X)$.

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- We want to find a homogeneous $\delta > 0$ such that $u(S)$ is $\delta$-far from weakly compact whenever $S \subset \mathbb{Q}$ is not rel. compact.
- We would color $S$ acording to the $\delta$ necessary. Do we have a Ramsey theorem that allows to homogenize? The one recently found by A. and Todorcevic does the job.

# Unconditional bases

**Question**

How to produce unconditional bases $B$ such that $\mathscr{R}(B)$ is Tukey equivalent to any of $\{0\}$, $\mathbb{N}$, $\mathbb{N}^{\mathbb{N}}$, $\mathscr{K}(\mathbb{Q})$, $Fin(\mathbb{R})$?

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It happens that $\mathbb{R}(B(\mathbb{A}_E)) \sim \mathscr{K}(E)$ so one applies Fremlin. From $E$ with $|E| = \omega_1$, one gets $B$ with $\mathscr{R}(B) \sim Fin(\omega_1)$