Banach spaces with rich $\mathcal{L}_\infty$ structure

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"Scalar-plus-Compact" property

- A separable space X is a $\mathcal{L}_\infty$ space, if there exists a constant $C > 0$ and an increasing sequence of finite dimensional spaces $(F_n)_n$ such that each F_n is C-isomorphic to $\ell_\infty(dimF_n)$ and $\overline{\cup_n F_n} = X$.
- (Lewis - Stegall) If X is a separable $\mathcal{L}_\infty$ space then $X^* \simeq \ell_1$ or $X^* \simeq M[0, 1]$.
- (Pelczynski) If X^* is non-separable then X isomorphically contains ℓ_1.

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Theorem (*S. A., R. Haydon 2011*)

There exists a $\mathcal{L}_\infty$ Hereditarily Indecomposable Banach space $\mathfrak{X}_K$ such that $\mathfrak{X}_K^*$ is isomorphic to $\ell_1(\mathbb{N})$ and every $T : \mathfrak{X}_K \to \mathfrak{X}_K$ is of the form $T = \lambda I + K$ with K a compact operator.

- This is the first example of a Banach space X such that every operator $T \in \mathcal{L}(X)$ admits a non trivial closed invariant subspace.

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Problem

Does there exist a separable Banach space $\mathfrak{X}$ with the hereditary "scalar-plus-compact" property? i.e. for every Y closed subspace of $\mathfrak{X}$ every operator $T \in \mathcal{L}(Y, \mathfrak{X})$ is of the form $T = \lambda I_{Y,\mathfrak{X}} + K$ with $K \in \mathcal{K}(Y, \mathfrak{X})$.

Problem (weaker version)

Does there exists $\mathfrak{X}$ saturated by infinite dimensional closed subspaces each one satisfying the "scalar-plus-compact" property?

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Problem (Ramsey Property for "$s + c$")

Let $\mathfrak{X}$ be a separable Banach space. Does there exist an infinite dimensional closed subspace Y of $\mathfrak{X}$ such that either:

- Every further subspace of Y satisfies the "scalar-plus-compact" property.

  or

- Every infinite dimensional closed subspace of Y does not satisfy the "scalar-plus-compact" property.

Problem (weaker version)

Assuming that the space $\mathfrak{X}$ has a Schauder basis, what is the answer to the above for the class of all block subspaces of $\mathfrak{X}$.

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A reflexive space with ISP

Theorem (S.A, Pavlos Motakis, Proc.LMS (2014))

There exists a reflexive HI Banach space $\mathfrak{X}_{ISP}$ satisfying the following:

(i) For every Y closed subspace of $\mathfrak{X}_{ISP}$, every $T \in \mathcal{L}(Y)$ is of the form $T = \lambda I + S$, with S strictly singular.

(ii) For every Y and $Q, S, T : Y \to Y$ strictly singular operators, the composition QST is a compact one.

- Every Banach space satisfying (i) and (ii), also satisfies the hereditary Invariant Subspace Property (i.e. for every infinite dimensional closed subspace Y of $\mathfrak{X}_{ISP}$ and every operator $T \in \mathcal{L}(Y)$ admits a non trivial closed invariant subspace).

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Rosenthal's Problem

- A Banach space X is said to be $\mathcal{L}_\infty$ saturated if every closed infinite dimension subspace Y of X contains a further subspace which is an $\mathcal{L}_\infty$ space.
- The known examples of $\mathcal{L}_\infty$ saturated Banach spaces are the classical ones, namely the space c_0 and the spaces $C(\alpha)$ where α is an infinite ordinal number with the usual order topology.

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Does there exist a Banach space X which is $\mathcal{L}_\infty$ saturated and does not contain a subspace isomorphic to c_0 ?

- The problem was stated by H.P. Rosenthal in the 80's and in particular he was interested for a negative answer.
- If such a space exists then passing to a subspace we may assume that a $\mathcal{L}_\infty$ space exists which is $\mathcal{L}_\infty$ saturated and does not contain c_0.
- Any $\mathcal{L}_\infty$ space which is $\mathcal{L}_\infty$ saturated has separable dual. Moreover, if it does not contain c_0 then it does not contain unconditional basic sequences. Hence, by Gowers Dichotomy it contains an HI subspace.

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- A solution to the Rosenthal's Problem will yield answers to some of the problems stated before.
- The aim of the present talk is to present some recent joint work with Pavlos Motakis related to Rosenthal's problem.

Theorem

There exists a $\mathcal{L}_\infty$ space $\mathfrak{X}$ with a basis $(d_n)_n$ such that for every infinite subset L of $\mathbb{N}$ the subspace $\mathfrak{X}_L = \overline{< d_n : n \in L >}$ contains a $\mathcal{L}_\infty$ space.

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General Bourgain-Delbaen construction

- A BD $\mathcal{L}_\infty$ space is a separable subspace $\mathfrak{X}$ of $\ell_\infty(\Gamma)$ with the supremum norm.
- The set Γ is countable, $\Gamma = \cup_{n=1}^{\infty}\Gamma_n$ where (Γ_n) is an increasing sequence of finite sets.
- We set $\Delta_1 = \Gamma_1$ and $\Delta_{n+1} = \Gamma_{n+1} \setminus \Gamma_n$.
- There exists $C > 0$ and extension operators

$$i_n : \ell_\infty(\Gamma_n) \to \ell_\infty(\Gamma)$$

(i.e. $i_n(x)|_{\Gamma_n} = x$) such that $\|i_n\| \leqslant C$ for every $n \in \mathbb{N}$.

- Hence i_n is C- isomorphic embedding for every $n \in \mathbb{N}$.

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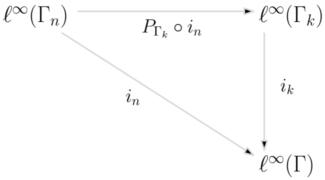

- For $\gamma \in \Delta_q$ we set $d_\gamma = i_q(e_\gamma)$ and
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$$\mathfrak{X} = \overline{\langle \{ d_\gamma : \gamma \in \Gamma \} \rangle}$$

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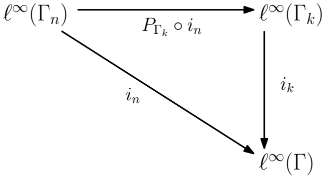

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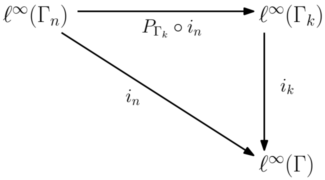

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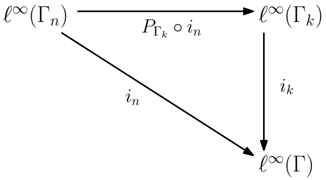

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- For every $\gamma \in \Delta_{q+1}$ we define a linear functional

$$c^*_\gamma = e^*_\gamma \circ i_q : \ell_\infty(\Gamma_q) \to \mathbb{R}.$$

- For every $p \leqslant q$ and $x \in \ell_\infty(\Gamma_p)$

$$i_p(x)(\gamma) = c^*_\gamma(i_q(i_p(x)|_{\Gamma_q}))$$

- The above explains that the construction of specific BD spaces essentially concerns the definition of the functionals $c^*_\gamma, \gamma \in \Gamma$ which by induction determine the extension operators $i_q, q \in \mathbb{N}$.

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- For every $\gamma \in \Delta_{q+1}$ we define a linear functional

$$c_\gamma^* = e_\gamma^* \circ i_q : \ell_\infty(\Gamma_q) \to \mathbb{R}.$$

- For every $p \leqslant q$ and $x \in \ell_\infty(\Gamma_p)$

$$i_p(x)(\gamma) = c_\gamma^*(i_q(i_p(x)|_{\Gamma_q}))$$

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Properties of the space $\mathfrak{X}$

- The space $\mathfrak{X}$ is a $C-\mathcal{L}_\infty$ space.

  Indeed, for $q\in\mathbb{N}$ $\langle\{d_\gamma|_{\Gamma_q}:\ \gamma\in\Gamma_q\}\rangle=\ell_\infty(\Gamma_q)$.

- Setting $F_q=i_q[\ell_\infty(\Delta_q)]$, $(F_q)_q$ is an *FDD* for the space $\mathfrak{X}$.
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- We denote with e^*_γ the evaluation functional restricted on $\mathfrak{X}$. Then the family $\{e^*_\gamma : \gamma \in \Gamma\}$ is equivalent to ℓ_1 basis. Moreover if $\{d_\gamma : \gamma \in \Gamma\}$ is shrinking, then $\{e^*_\gamma : \gamma \in \Gamma\}$ generates the dual of $\mathfrak{X}$.

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The double role of e_γ^*

- For $\gamma \in \Gamma_{n+1} \setminus \Gamma_n$, $e_\gamma^* = d_\gamma^* + c_\gamma^*$ where

$$c_\gamma^* : \ell_\infty(\Gamma_n) \to \mathbb{R}$$

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Mixed Tsirelson $\mathcal{L}_\infty$ spaces

- We define the functionals $\{c_\gamma^* : \gamma \in \Gamma\}$ in a similar manner as the functionals in the norming set of a Mixed Tsirelson spaces.
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- There are three types of functionals.
- Let $\gamma \in \Delta_{q+1}$.

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$$c^*_\gamma = 0$$

Then $w(\gamma) = 0,\ a(\gamma) = 0$.

(2)

$$c^*_\gamma = \frac{1}{m_j} e^*_\xi \circ P_I$$

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The evaluation analysis of e^*_γ

- For $\gamma \in \Delta_{q+1}$ with $w(\gamma) = \frac{1}{m_j}$ and $\alpha(\gamma) = k \leqslant n_j$, there exist $\xi_1, \xi_2, \ldots, \xi_k, \eta_1, \eta_2, \ \ldots, \eta_k$ and $I_1, I_2, \ldots, I_k$ such that

$$e^*_\gamma = \frac{1}{m_j} \sum_{i=1}^{k} e^*_{\xi_i} \circ P_{I_i} + \sum_{i=1}^{k} d^*_{\eta_i}.$$

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- The mixed Tsirelson BD-$\mathcal{L}_\infty$ space is defined by induction by setting $\Delta_1 = \Gamma_1 = \{\gamma_1\}$ with $w(\gamma_1)$.
- Assuming that $\Delta_1, \ldots, \Delta_q$ have been defined, Δ_{q+1} is defined by taking all possible $\phi \in \ell_1(\Gamma_q)$ of the previous form with $w(\phi) = m_j$, $j \leqslant q+1$.
- It is shown that the extensions operators $i_q : \ell_\infty(\Gamma_q) \to \ell_\infty(\Gamma)$ satisfy $\|i_q\| \leqslant 1 + 8/m_1$.

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Theorem

The mixed Tsirelson BD-$\mathcal{L}_\infty$ space $\mathfrak{X} =$ has separable dual and does not contain c_0 or ℓ_p for $1 < p < \infty$. Every skipped block sequence with respect to the FDD $\{i_q(\ell_\infty(\Delta_q))\}_{n=0}^\infty$ generates a reflexive subspace. Hence, every skipped block sequence does not generate a $\mathcal{L}_\infty$ space.

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Theorem

There exists a $\mathcal{L}_\infty$ space $\mathfrak{X}$ with a basis $(d_n)_n$ such that for every infinite subset L of $\mathbb{N}$ the subspace $\mathfrak{X}_L = \overline{< d_n : n \in L >}$ contains a $\mathcal{L}_\infty$ space.

Theorem (A first Step)

There exists a Mixed Tsirelson BD $\mathcal{L}_\infty$ space not containing c_0 or ℓ_p for $1 \leqslant p < \infty$ and a skipped block basis that that generates a $\mathcal{L}_\infty$ space.

- The definition of this space requires a modification of the previous techniques.

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- How to create a BD $\mathcal{L}_\infty$ space with a Schauder basis $(d_n)_n$ and a block sequence $(x_k)_k$ such that $\overline{< x_k : k \in \mathbb{N} >}$ is also a $\mathcal{L}_\infty$ space.
- We start with a sequence of parameters $(m_j, n_j)_j$ as before.
- For define a set $\Gamma = \cup_{n=1}^{\infty} \Delta_q$ such that each $\Delta_q = \{\gamma\}$ and if $\gamma \in \Delta_{2q-1}$ then

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- The idea in the use of the multiple weights is to succeed that the γ which codes c_γ^*, at the same time norms a block vector.
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- We inductively define $\theta_1, \eta_1, \theta_2, \eta_2, \ldots, \theta_{n_{2j_{\ell+1}-1}}, \eta_{n_{2j_{\ell+1}-1}}$ with $w(\theta_i) = 0$ for all i and
- $c^*_{\eta_i} = (1/n_{2j_{\ell+1}-1}) \left(e^*_\gamma + \frac{1}{m_{2i}} e^*_\xi \circ P_I \right) + e^*_{\eta_{i-1}} + (1/m_{2j_{\ell+1}-1}) e^*_{\theta_i}$.
- The $\eta_{n_{2j_{\ell+1}-1}} = \eta$ norms the following vector:

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- The construction yields that the family

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- defines a skipped block sequence and the space generated by this sequence is normed by the multiple weight γ's.
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