On hereditary approximation property

Razvan Anisca and Christopher Chlebovec

Lakehead University, Canada

First Brazilian Workshop in Geometry of Banach spaces

August 2014

Definition A Banach space X has the *approximation property* provided that, for every compact set $K \subset X$ and every $\epsilon > 0$, there exists a finite rank operator $T : X \to X$ such that $\|Tz - z\| < \epsilon$ for every $z \in K$.

We say that a Banach space X has the *hereditary approximation property* (HAP) if all its subspaces have the approximation property.

Definition A Banach space X has the *approximation property* provided that, for every compact set $K \subset X$ and every $\epsilon > 0$, there exists a finite rank operator $T : X \to X$ such that $\|Tz - z\| < \epsilon$ for every $z \in K$.

We say that a Banach space X has the *hereditary approximation property* (HAP) if all its subspaces have the approximation property.

Examples of spaces with HAP:

Definition A Banach space X has the *approximation property* provided that, for every compact set $K \subset X$ and every $\epsilon > 0$, there exists a finite rank operator $T : X \to X$ such that $\|Tz - z\| < \epsilon$ for every $z \in K$.

We say that a Banach space X has the *hereditary approximation property* (HAP) if all its subspaces have the approximation property.

Examples of spaces with HAP:

- Johnson (1980) : for suitable $k_n \uparrow \infty$ and $p_n \to 2$, the space $X = \left(\sum_n \oplus \ell_{p_n}^{k_n} \right)_{\ell_2}$ has HAP and is not isomorphic to ℓ_2
- Johnson (1980): the 2-convexified Tsirelson's space $T^{(2)}$

- Pisier (1988): weak Hilbert spaces have HAP

- Pisier (1988): weak Hilbert spaces have HAP

Definition (Pisier) A Banach space X is a *weak Hilbert space* provided there exist $\delta > 0$ and $K \geq 1$ such that: every finite-dimensional subspace $E \subseteq X$ contains a further subspace F with $\dim F \geq \delta \dim E$ and

$$d\left(F, \ell_2^{\dim F}\right) \leq K \quad \text{and} \quad \|P : X \longrightarrow F\| \leq K.$$

- Pisier (1988): weak Hilbert spaces have HAP

Definition (Pisier) A Banach space X is a *weak Hilbert space* provided there exist $\delta > 0$ and $K \geq 1$ such that: every finite-dimensional subspace $E \subseteq X$ contains a further subspace F with $\dim F \geq \delta \dim E$ and

$$d\left(F, \ell_2^{\dim F}\right) \leq K \quad \text{and} \quad \|P : X \longrightarrow F\| \leq K.$$

It is known that weak Hilbert spaces satisfy some very strong types of approximation properties, in particular they admit a finite-dimensional Schauder decomposition (Maurey-Pisier)

Question: Does every weak Hilbert space have a Schauder basis?

Spaces which do not have HAP

Szankowski (1978): a Banach space X does not have HAP (i.e., admits a subspace *without* the approximation property) whenever

$$p(X) = \sup\{p : X \text{ has type } p\} < 2$$

or

$$q(X) = \inf\{q : X \text{ has cotype } q\} > 2.$$

Spaces which do not have HAP

Szankowski (1978): a Banach space X does not have HAP (i.e., admits a subspace *without* the approximation property) whenever

$$p(X) = \sup\{p : X \text{ has type } p\} < 2$$

or

$$q(X) = \inf\{q : X \text{ has cotype } q\} > 2.$$

X has *type* p (≤ 2) if there exists $C > 0$ such that for all n and $x_1, \ldots, x_n \in X$

$$\left(\int_0^1 \left\|\sum_{i=1}^n r_i(t)x_i\right\|^2 dt\right)^{1/2} \leq C\,(\sum_{i=1}^n \|x_i\|^p)^{1/p}$$

New result on HAP

(Johnson and Szankowski, Annals of Mathematics 2012)

New result on HAP

(Johnson and Szankowski, Annals of Mathematics 2012)

Definition For a Banach space X the *sequence of distances to a Hilbert space* $\{d_n(X)\}$ is defined by

$$d_n(X) = \sup\{d(E, l_2^{\dim E}) \mid E \subset X,\ \dim E \leq n\}.$$

New result on HAP

(Johnson and Szankowski, Annals of Mathematics 2012)

Definition For a Banach space X the *sequence of distances to a Hilbert space* $\{d_n(X)\}$ is defined by

$$d_n(X) = \sup\{d(E, l_2^{\dim E}) \mid E \subset X,\ \dim E \leq n\}.$$

Theorem *If X is a Banach space such that the sequence $\{d_n(X)\}$ grows sufficiently slow as $n \to \infty$, then X must have HAP.*

New result on HAP

(Johnson and Szankowski, Annals of Mathematics 2012)

Definition For a Banach space X the *sequence of distances to a Hilbert space* $\{d_n(X)\}$ is defined by

$$d_n(X) = \sup\{d(E, l_2^{\dim E}) \mid E \subset X,\ \dim E \leq n\}.$$

Theorem *If X is a Banach space such that the sequence $\{d_n(X)\}$ grows sufficiently slow as $n \to \infty$, then X must have HAP.*

The rate of growth of $\{d_n(X)\}$ needed is of (inverse) Ackermann type: for a fixed $A > 1$, define $D : \mathbb{N} \to \mathbb{N}$ by $D(j) = 3[A^j]$ and let

$$\gamma(j) = \underbrace{D \circ D \circ \ldots \circ D}_{3^j+1}(1).$$

The theorem requires $d_{\gamma(j)}(X) = o\left(\beta^{-j}\right)$, for some $\beta < 1$.

Question (J-S) *If $d_n(X) = o(\log n)$, does it imply that X has HAP?*

Question (J-S) *If $d_n(X) = o(\log n)$, does it imply that X has HAP?*

Our result

When the rate of growth of $d_n(X)$ is at least the same as $(\log n)^\beta$, for some $\beta > 1$, then X does not necessarily have HAP.

Question (J-S) *If $d_n(X) = o(\log n)$, does it imply that X has HAP?*

Our result

When the rate of growth of $d_n(X)$ is at least the same as $(\log n)^\beta$, for some $\beta > 1$, then X does not necessarily have HAP.

Theorem (A-Chlebovec) *Let $\mathcal{X}$ be the class of Banach spaces X which have cotype 2 and are of the form $X = \ell_2(Z)$, for some Banach space Z. Let X in $\mathcal{X}$ and assume that there exist constants $\alpha > 0, \beta > 1$ such that*

$$d_n(X) \geq \alpha\,(\log n)^\beta \qquad \forall n.$$

Then X does not have HAP.

Criterion (Enflo): *Assume that in a Banach space Z there exist bounded sequences $\{z_k\}_k \subset Z$, $\{z_k^*\}_k \subset Z^*$ such that*
(i) $z_k^*(z_k) = 1$, for all k, and $z_k^* \xrightarrow{w*} 0$.
(ii) for every linear operator $T : Z \to Z$ and n we have

$$|\beta_n(T) - \beta_{n-1}(T)| \leq \|T\|\alpha_n$$

with $\sum_n \alpha_n < \infty$, where

$$\beta_n(T) = \frac{1}{2^n} \sum_{2^n \leq k < 2^{n+1}} z_k^*(Tz_k)$$

Then Z does not have the bounded approximation property.

Criterion (Enflo): *Assume that in a Banach space Z there exist bounded sequences $\{z_k\}_k \subset Z$, $\{z_k^*\}_k \subset Z^*$ such that*
(i) $z_k^*(z_k) = 1$, for all k, and $z_k^* \xrightarrow{w*} 0$.
(ii) for every linear operator $T : Z \to Z$ and n we have

$$|\beta_n(T) - \beta_{n-1}(T)| \leq \|T\|\alpha_n$$

with $\sum_n \alpha_n < \infty$, where

$$\beta_n(T) = \frac{1}{2^n} \sum_{2^n \leq k < 2^{n+1}} z_k^*(Tz_k)$$

Then Z does not have the bounded approximation property.

Proof $\beta_j(I) = 1$ for all j and $\beta_j(T) \to 0$ if T has finite rank. Then, for any T of finite rank

$$\|I - T\|_{[z_k]_{2^n \leq k < 2^{n+1}}} \geq |\beta_n(I - T)| \geq 1 - |\beta_n(T)|$$

$$\geq 1 - \sum_{j=n}^{\infty} |\beta_{j+1}(T) - \beta_j(T)| \geq 1 - \|T\| \sum_{j=n}^{\infty} \alpha_j.$$

Some useful vectors in $\ell_2(X)$:

Some useful vectors in $\ell_2(X)$:

Definition Let X be a Banach space. For $n = 1, 2, \ldots$, let $\kappa_n(X) \geq 1$ be the smallest constant κ such that for every 1-unconditional normalized sequence of vectors $\{u_i\}_{i=1}^{l}$ in X, with $1 \leq l \leq n$, one has

$$\kappa^{-1}\sqrt{l} \leq \|\sum_{i=1}^{l} u_i\| \leq \kappa\sqrt{l}$$

[We say that X has *property* (H) provided $\kappa(X) := \sup_n \kappa_n(X) < \infty$].

Some useful vectors in $\ell_2(X)$:

Definition Let X be a Banach space. For $n = 1, 2, \ldots$, let $\kappa_n(X) \geq 1$ be the smallest constant κ such that for every 1-unconditional normalized sequence of vectors $\{u_i\}_{i=1}^{l}$ in X, with $1 \leq l \leq n$, one has

$$\kappa^{-1}\sqrt{l} \leq \| \sum_{i=1}^{l} u_i \| \leq \kappa\sqrt{l}$$

[We say that X has *property* (H) provided $\kappa(X) := \sup_n \kappa_n(X) < \infty$].

Proposition (Nielsen-Tomczak J.) *Let X be a Banach space which has type 2. There is a universal constant $C \geq 1$ such that, for all n,*

$$d_n(X) \leq C T_2(X)^3 \kappa_n(\ell_2(X)) \,.$$

This means: for all $n \geq 1$ there are 1-unconditional normalized vectors $\{z_1, \ldots, z_n\} \subset \ell_2(X)$ such that either

$$\| \sum_{i=1}^{n} z_i \|_{\ell_2(X)} > c d_n(X) n^{1/2}$$

or

$$\| \sum_{i=1}^{n} z_i \|_{\ell_2(X)} < \frac{1}{c d_n(X)} n^{1/2}.$$

When X has type 2 the first alternative fails.

This means: for all $n \geq 1$ there are 1-unconditional normalized vectors $\{z_1, \ldots, z_n\} \subset \ell_2(X)$ such that either

$$\|\sum_{i=1}^{n} z_i\|_{\ell_2(X)} > c d_n(X) n^{1/2}$$

or

$$\|\sum_{i=1}^{n} z_i\|_{\ell_2(X)} < \frac{1}{c d_n(X)} n^{1/2}.$$

When X has type 2 the first alternative fails.

Proposition *Let X be a Banach space which has type 2. There is $c = c(T_2(X)) > 0$ such that, for all $n \geq 1$, there exists a 1-unconditional normalized sequence of vectors $\{z_i\}_{i=1}^{n} \subset \ell_2(X)$ with*

$$\|\sum_{i=1}^{n} z_i\|_{\ell_2(X)} \leq \frac{1}{c d_n(X)} n^{1/2}.$$

Theorem *Let X be a Banach space which has cotype 2. Assume there exist constants $\alpha > 0$ and $\beta > 1$ such that*

$$d_n(X) \geq \alpha \left(\log(n)\right)^{\beta} \qquad \forall n.$$

Then $\ell_2(X)$ has a subspace without the approximation property.

Theorem *Let X be a Banach space which has cotype 2. Assume there exist constants $\alpha > 0$ and $\beta > 1$ such that*

$$d_n(X) \geq \alpha \, (\log(n))^\beta \qquad \forall n.$$

Then $\ell_2(X)$ has a subspace without the approximation property.

Sketch of the proof: WLOG we can assume that X^* has type 2. Then

$$d_n(X^*) \geq \alpha / T_2(X^*) \; (\log(n))^\beta \, .$$

Theorem *Let X be a Banach space which has cotype 2. Assume there exist constants $\alpha > 0$ and $\beta > 1$ such that*

$$d_n(X) \geq \alpha \left(\log(n)\right)^{\beta} \qquad \forall n.$$

Then $\ell_2(X)$ has a subspace without the approximation property.

Sketch of the proof: WLOG we can assume that X^* has type 2. Then

$$d_n(X^*) \geq \alpha / T_2(X^*) \; \left(\log(n)\right)^{\beta}.$$

For all n, we use (similarly as Szankowski) a partition ∇_n of $\{2^n, \ldots, 2^{n+1} - 1\}$. For $A \in \nabla_n$ we can find $\{e_i^*\}_{i \in A} \in \ell_2(X^*)$ such that

$$\left\| \sum_{i \in A} e_i^* \right\|_{\ell_2(X^*)} \leq \frac{c}{d_{|A|}(X^*)} |A|^{1/2}$$

Theorem *Let X be a Banach space which has cotype 2. Assume there exist constants $\alpha > 0$ and $\beta > 1$ such that*

$$d_n(X) \geq \alpha\, (\log(n))^\beta \qquad \forall n.$$

Then $\ell_2(X)$ has a subspace without the approximation property.

Sketch of the proof: WLOG we can assume that X^* has type 2. Then

$$d_n(X^*) \geq \alpha / T_2(X^*)\ (\log(n))^\beta .$$

For all n, we use (similarly as Szankowski) a partition ∇_n of $\{2^n, \ldots, 2^{n+1} - 1\}$. For $A \in \nabla_n$ we can find $\{e_i^*\}_{i \in A} \in \ell_2(X^*)$ such that

$$\left\| \sum_{i \in A} e_i^* \right\|_{\ell_2(X^*)} \leq \frac{c}{d_{|A|}(X^*)} |A|^{1/2}$$

Let $\{e_j\}_{j \in A} \subset \ell_2(X)$ such that $e_i^*(e_j) = \delta_{ij}$. Denote $X_A = \mathrm{span}\{e_i\}_{i \in A}$ and then set

$$Y = \left(\sum_{n=1}^{\infty} \sum_{A \in \nabla_n} \bigoplus X_A \right)_{\ell_2} \subset \ell_2(X).$$

$$Y = \left(\sum_{n=1}^{\infty} \sum_{A \in \nabla_n} \bigoplus X_A \right)_{\ell_2} \subset \ell_2(X)$$

with basis vectors $f_i = (0, \ldots, 0, e_i, 0, \ldots) \in Y$ ($e_i \in X_A$ in its corresponding position for $i \in A$)

$$Y^* = \left(\sum_{n=1}^{\infty} \sum_{A \in \nabla_n} \bigoplus X_A^* \right)_{\ell_2}$$

with basis vectors $f_i^* = (\mathbf{0}, \ldots, \mathbf{0}, e_i^*\Big|_{X_A}, \mathbf{0}, \ldots)$.

$$Y = \left(\sum_{n=1}^{\infty} \sum_{A \in \nabla_n} \bigoplus X_A \right)_{\ell_2} \subset \ell_2(X)$$

with basis vectors $f_i = (0, \ldots, 0, e_i, 0, \ldots) \in Y$ ($e_i \in X_A$ in its corresponding position for $i \in A$)

$$Y^* = \left(\sum_{n=1}^{\infty} \sum_{A \in \nabla_n} \bigoplus X_A^* \right)_{\ell_2}$$

with basis vectors $f_i^* = (\mathbf{0}, \ldots, \mathbf{0}, e_i^*\Big|_{X_A}, \mathbf{0}, \ldots)$.

Subspace without AP : $Z = \overline{\text{span}}\{z_i\}_i \subset Y$

$$z_i = f_{2i} - f_{2i+1} + f_{4i} + f_{4i+1} + f_{4i+2} + f_{4i+3}$$

It remains to check Enflo's criterion for $\{z_i\}_i, \{z_i^*\}$

$$z_i^* = \frac{1}{2}(f_{2i}^* - f_{2i+1}^*)\Big|_Z = \frac{1}{4}(f_{4i}^* + f_{4i+1}^* + f_{4i+2}^* + f_{4i+3}^*)\Big|_Z$$

Criterion (Enflo): *Assume that in Z there exist bounded sequences $\{z_i\}_i \subset Z$, $\{z_i^*\}_i \subset Z^*$ s. t.*

(i) $z_i^*(z_i) = 1$, *for all* i, *and* $z_i^* \xrightarrow{w*} 0$.

(ii) for every linear operator $T : Z \to Z$ *and* n *we have*

$$|\beta_n(T) - \beta_{n-1}(T)| \leq \|T\|\alpha_n$$

with $\sum_n \alpha_n < \infty$, *where*

$$\beta_n(T) = \frac{1}{2^n} \sum_{2^n \leq i < 2^{n+1}} z_i^*(Tz_i)$$

Then Z does not have the bounded approximation property.

$$\begin{cases} z_i = f_{2i} - f_{2i+1} + f_{4i} + f_{4i+1} + f_{4i+2} + f_{4i+3} \\ z_i^* = \frac{1}{2}(f_{2i}^* - f_{2i+1}^*)\Big|_Z = \frac{1}{4}(f_{4i}^* + \ldots + f_{4i+3}^*)\Big|_Z \\ \beta_n(T) = \frac{1}{2^n} \sum\limits_{2^n \le i < 2^{n+1}} z_i^*(Tz_i) \end{cases}$$

$$\begin{cases} z_i = f_{2i} - f_{2i+1} + f_{4i} + f_{4i+1} + f_{4i+2} + f_{4i+3} \\ z_i^* = \frac{1}{2}(f_{2i}^* - f_{2i+1}^*)\Big|_Z = \frac{1}{4}(f_{4i}^* + \ldots + f_{4i+3}^*)\Big|_Z \\ \beta_n(T) = \frac{1}{2^n} \sum_{2^n \le i < 2^{n+1}} z_i^*(Tz_i) \end{cases}$$

- $\beta_n(T) - \beta_{n-1}(T) = \frac{1}{2^{n+1}} \sum_{2^{n+1} \le i < 2^{n+2}} f_i^*(Ty_i),$

$y_i =$ a linear combination (with bounded coefficients) of 9 vector basis f_j <u>not</u> including f_i

$$
\begin{cases}
z_i = f_{2i} - f_{2i+1} + f_{4i} + f_{4i+1} + f_{4i+2} + f_{4i+3} \\
z_i^* = \frac{1}{2}(f_{2i}^* - f_{2i+1}^*)\Big|_Z = \frac{1}{4}(f_{4i}^* + \ldots + f_{4i+3}^*)\Big|_Z \\
\beta_n(T) = \frac{1}{2^n} \sum_{2^n \le i < 2^{n+1}} z_i^*(Tz_i)
\end{cases}
$$

- $$\beta_n(T) - \beta_{n-1}(T) = \frac{1}{2^{n+1}} \sum_{2^{n+1} \le i < 2^{n+2}} f_i^*(Ty_i),$$

$y_i =$ a linear combination (with bounded coefficients) of 9 vector basis f_j <u>not</u> including f_i

- $$|\beta_n(T) - \beta_{n-1}(T)| \le \frac{1}{2^{n+1}} \sum_{A \in \nabla_{n+1}} |\sum_{i \in A} f_i^*(Ty_i)|$$

$$= \frac{1}{2^{n+1}} \sum_{A \in \nabla_{n+1}} \left| \int_0^1 \left(\sum_{i \in A} r_i(t) f_i^* \right) \left(\sum_{i \in A} r_i(t) Ty_i \right) dt \right|$$

$$\leq \frac{1}{2^{n+1}} \sum_{A \in \nabla_{n+1}} \max_{\varepsilon_i = \pm 1} \left\{ \left\| \sum_{i \in A} \varepsilon_i f_i^* \right\| \left\| T\left(\sum_{i \in A} \varepsilon_i y_i \right) \right\| \right\}$$

$$\leq \frac{1}{2^{n+1}} \frac{2^{n+1}}{m_{n+1}} \|T\| \max_{A, \varepsilon_i = \pm 1} \left\| \sum_{i \in A} \varepsilon_i f_i^* \right\| \cdot \max_{A, \varepsilon_i = \pm 1} \left\| \sum_{i \in A} \varepsilon_i y_i \right\|$$

(here we used $|A| = m_{n+1} = 2^{(n+1)/8}$)

$$\leq \frac{1}{2^{n+1}} \sum_{A\in\nabla_{n+1}} \max_{\varepsilon_i=\pm 1} \left\{ \left\| \sum_{i\in A} \varepsilon_i f_i^* \right\| \left\| T\left(\sum_{i\in A} \varepsilon_i y_i \right) \right\| \right\}$$

$$\leq \frac{1}{2^{n+1}} \frac{2^{n+1}}{m_{n+1}} \|T\| \max_{A,\varepsilon_i=\pm 1} \left\| \sum_{i\in A} \varepsilon_i f_i^* \right\| \cdot \max_{A,\varepsilon_i=\pm 1} \left\| \sum_{i\in A} \varepsilon_i y_i \right\|$$

(here we used $|A| = m_{n+1} = 2^{(n+1)/8}$)

$$Y = \left(\sum_{n=1}^{\infty} \sum_{A\in\nabla_n} \bigoplus X_A \right)_{\ell_2}, \quad Y^* = \left(\sum_{n=1}^{\infty} \sum_{A\in\nabla_n} \bigoplus X_A^* \right)_{\ell_2}$$

$$\leq \frac{1}{2^{n+1}} \sum_{A \in \nabla_{n+1}} \max_{\varepsilon_i = \pm 1} \left\{ \left\| \sum_{i \in A} \varepsilon_i f_i^* \right\| \left\| T \left(\sum_{i \in A} \varepsilon_i y_i \right) \right\| \right\}$$

$$\leq \frac{1}{2^{n+1}} \frac{2^{n+1}}{m_{n+1}} \|T\| \max_{A, \varepsilon_i = \pm 1} \left\| \sum_{i \in A} \varepsilon_i f_i^* \right\| \cdot \max_{A, \varepsilon_i = \pm 1} \left\| \sum_{i \in A} \varepsilon_i y_i \right\|$$

(here we used $|A| = m_{n+1} = 2^{(n+1)/8}$)

$$Y = \left(\sum_{n=1}^{\infty} \sum_{A \in \nabla_n} \bigoplus X_A \right)_{\ell_2}, \ Y^* = \left(\sum_{n=1}^{\infty} \sum_{A \in \nabla_n} \bigoplus X_A^* \right)_{\ell_2}$$

- $\left\| \sum_{i \in A} \varepsilon_i f_i^* \right\| \leq \frac{c}{d_{|A|}(X^*)} |A|^{1/2} = \frac{c}{(\log(m_{n+1}))^{\beta}} m_{n+1}^{1/2}$

$$\leq \frac{1}{2^{n+1}} \sum_{A \in \nabla_{n+1}} \max_{\varepsilon_i = \pm 1} \left\{ \left\| \sum_{i \in A} \varepsilon_i f_i^* \right\| \left\| T\left(\sum_{i \in A} \varepsilon_i y_i \right) \right\| \right\}$$

$$\leq \frac{1}{2^{n+1}} \frac{2^{n+1}}{m_{n+1}} \|T\| \max_{A, \varepsilon_i = \pm 1} \left\| \sum_{i \in A} \varepsilon_i f_i^* \right\| \cdot \max_{A, \varepsilon_i = \pm 1} \left\| \sum_{i \in A} \varepsilon_i y_i \right\|$$

(here we used $|A| = m_{n+1} = 2^{(n+1)/8}$)

$$Y = \left(\sum_{n=1}^{\infty} \sum_{A \in \nabla_n} \bigoplus X_A \right)_{\ell_2}, \; Y^* = \left(\sum_{n=1}^{\infty} \sum_{A \in \nabla_n} \bigoplus X_A^* \right)_{\ell_2}$$

- $\left\| \sum_{i \in A} \varepsilon_i f_i^* \right\| \leq \frac{c}{d_{|A|}(X^*)} |A|^{1/2} = \frac{c}{(\log(m_{n+1}))^{\beta}} m_{n+1}^{1/2}$
- special choice of $A's$: $\left\| \sum_{i \in A} \varepsilon_i y_i \right\| \leq 9|A|^{1/2} = 9 m_{n+1}^{1/2}$

($y_i =$ a linear combination of 9 vector basis f_j)

Thus

$$|\beta_n(T) - \beta_{n-1}(T)| \leq c\|T\|\frac{1}{(\log(m_{n+1}))^{\beta}}$$

with $m_{n+1} = 2^{(n+1)/8}$.

Since $\beta > 1$, Enflo's criterion is satisfied:

$$\sum_n \frac{1}{(\log(m_{n+1}))^{\beta}} < \infty.$$

□

Final remarks

The result suggests a possible way of dealing with another question raised by Johnson and Szankoswski:

Question (J-S) Is HAP preserved under ℓ_2-sums?

Final remarks

The result suggests a possible way of dealing with another question raised by Johnson and Szankoswski:

Question (J-S) Is HAP preserved under ℓ_2-sums?

Possible candidate for a counterexample:

$$X = \left(\sum_n \oplus \ell_{p_n}^{k_n} \right)_{\ell_2}$$

The parameters $k_n \uparrow \infty$ and $p_n \to 2$ can be chosen such that X has HAP and

$$d_n(X) \geq \alpha \left(\log \log \log(n) \right)^{\beta}$$

for some absolute constants $\alpha > 0$ and $\beta > 1$.

Final remarks

The result suggests a possible way of dealing with another question raised by Johnson and Szankoswski:

Question (J-S) Is HAP preserved under ℓ_2-sums?

Possible candidate for a counterexample:

$$X = \left(\sum_n \oplus \ell_{p_n}^{k_n} \right)_{\ell_2}$$

The parameters $k_n \uparrow \infty$ and $p_n \to 2$ can be chosen such that X has HAP and

$$d_n(X) \geq \alpha \left(\log \log \log(n) \right)^{\beta}$$

for some absolute constants $\alpha > 0$ and $\beta > 1$.

Question Is it true that $\ell_2(X)$ does not have HAP?