UPPER SEMICONTINUITY OF ATTRACTORS FOR A HYPERBOLIC EQUATION *

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1 Introduction

The large time behavior of solutions of the linearly damped wave equation

$$u_{tt} + 2\alpha u_t = (a_\nu(x)u_x)_x + f(u), 0 < x < 1, t > 0 \tag{1.1}$$

with Neumann boundary condition

$$u_x(0,t) = u_x(1,t) = 0, t > 0, \tag{1.2}$$

has been considered by several authors and some results showing almost no spatial dependence on the x variable were obtained - see, for example, [Carvalho], [Sola-Morales and Valencia], ... (see also [Conway, Hoff and Smoller], [Hale], [Fusco] for the equivalent problem for reaction-diffusion equations). As it is shown in [Carvalho], if $\inf\{a(x) : x \in [0,1]\}$ is sufficiently large, then any solution converges to a spatially homogeneous solution of (**??**) (in the case a constant, a similar result was obtained by [Sola-Morales and Valencia]).

For the wave equation with large damping, the existence of a exponentially attracting finite dimensional invariant manifold was shown by [Mora]. In this paper we consider the case in which a_ν is large except in a neighborhood of a fixed point x_1 in $[0,1]$, where it becomes small.

Since a_ν is large outside a small neighborhood of x_1, we expect that the solutions of (**??**) converge to a constant on each subinterval as $\nu \to 0+$. In fact, we will prove that

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if α is sufficiently large then, as $\nu \to 0$, the solutions of (**??**) approach the solutions of a system of O.D.Es.

Diffusion coefficients of this type have been considered by some authors (e.g., [Fusco], [Hale], [Carvalho and Pereira], [Carvalho and Oliveira]) in the context of parabolic equations and some nice results were obtained concerning to the structure of the global attractor.

The main goal of the present paper is to extend those results for equation (1.1), (1.2). Specifically, we will show that the global attractor $\mathcal{A}_\nu$ of (1.1), (1.2) approaches the attractor $\mathcal{A}_0$ of a system of two ordinary differential equations.

Remark 1.1 *In fact, one can consider the case where the diffusion coefficient becomes small in a neighborhood of a finite set of points. The difficulties thus introduced are only of a notational nature. We have chosen to keep the notation as simple as possible in order to better convey the main idea.*

Remark 1.2 *The case where the diffusion coefficient becomes large everywhere is not explicitly considered here. However, it will become clear from our approach that, in this case the asymptotic behavior is governed by a single first order differential equation. This improves the results of [Carvalho], where a* second *order equation is obtained.*

2 Hypotheses

In this section we state the hypotheses to be used throughout the paper.

H1 $f : \mathbb{R} \to \mathbb{R}$ is C^2 and satisfies the dissipativeness condition

$$\limsup_{|u|\to\infty} \frac{f(u)}{u} \leq -r, \tag{2.3}$$

for some $r > 0$.

H2 There exist $x_1 \in (0,1)$, positive constants ℓ_1, a_1, e_1, e_2 and functions $\ell_1' > \ell_1$, $a_1' > a_1$ of a small parameter $\nu > 0$ such that $\ell_1'(\nu) - \ell_1 = O(\nu^q)$, for some $0 < q < 1$, and $a_1'(\nu) - a_1 = o(1)$ as $\nu \to 0+$, in such a way that $a = a_\nu$ is a C^1 function satisfying

$$\begin{cases} a(x) \geq \frac{e_1}{\nu}, & \text{for} \quad 0 \leq x \leq x_1 - \nu\ell_1' \\ a(x) \geq \frac{e_2}{\nu}, & \text{for} \quad x_1 + \nu\ell_1' \leq x \leq 1 \\ a(x) \geq \nu a_1, & \text{for} \quad x_1 - \nu\ell_1' \leq x \leq x_1 + \nu\ell_1' \\ a(x) \leq \nu a_1', & \text{for} \quad x_1 - \nu\ell_1 \leq x \leq x_1 + \nu\ell_1 \end{cases} \tag{2.4}$$

3 The linear problem

Let X be the real Hilbert space of square integrable functions on $[0, 1]$. For each $0 < \nu \le \nu_0$, define the operator A_ν in X by

$$D(A_\nu) = \{u \in H^2(0,1) : u'(0) = u'(1) = 0\}$$

and

$$(A_\nu u)(x) = -(a_\nu(x)u'(x))',\ 0 < x < 1.$$

Since A_ν is a closed selfadjoint nonnegative operator in $L(X)$, A_ν is sectorial and therefore the fractional power spaces $X_\nu^\gamma = D(A_\nu^\gamma)$ endowed with the graph norm are well defined and they are Hilbert spaces. Since A_ν has compact resolvent, the spectrum of A_ν consists only of eigenvalues and, for each $0 < \gamma \le 1$, the embbeding $X_\nu^\gamma \subset X$ is compact. In this paper we shall assume that $\gamma = \frac{1}{2}$, in which case $X_\nu^{\frac{1}{2}} = H^1(0,1)$ endowed with the norm given by the inner product

$$\langle u, \tilde{u} \rangle = \int_0^1 [u(x)\tilde{u}(x) + a_\nu(x)u'(x)\tilde{u}'(x)]\, dx.$$

The following result is proved in [**?**].

Lemma 3.1 *There exists a constant $E > 0$ such that*

$$\sup_{0\le x\le 1} |\phi(x)| \le E \left(\int_0^1 [\phi(x)^2 + a_\nu(x)\phi'(x)^2\, dx \right)^{\frac{1}{2}},$$

for all $\nu > 0$ and $\phi \in H^1(0,1)$. Therefore, the constant appearing in the embbeding $L^\infty(0,1) \hookrightarrow X_\nu^{\frac{1}{2}}$ is independent of ν.

Let H be the Hilbert space $H^1(0,1) \times L^2(0,1)$ endowed with the norm defined by the inner product

$$\langle (u,v), (\tilde{u}, \tilde{v}) \rangle = \int_0^1 [u(x)\tilde{u}(x) + v(x)\tilde{(v)}(x) + a_\nu(x)u'(x)\tilde{u}'(x)]\, dx.$$

Let $C_\nu : D(C_\nu) \subset H \to H$ be defined by

$$D(C_\nu) = D(A_\nu) \times H^1(0,1)$$

and

$$C_\nu(u,v)(x) = (v(x), (a_\nu(x)u'(x))' - 2\alpha v(x)) \quad , \quad 0 < x < 1.$$

Lemma 3.2 *The operator C_ν is the generator of a strongly continuous semigroup on H.*

Proof. Let $B_\nu : D(B_\nu) = D(C_\nu) \to H$ be defined by

$$B_\nu(u,v) = (v, (a_\nu u')' - 2\alpha v - u).$$

If $(u,v) \in D(C_\nu)$, then integration by parts gives

$$\begin{aligned}\langle B_\nu(u,v),(u,v)\rangle &= \int_0^1 (uv + au'v' + v(au')' - uv - 2\alpha v^2)\,dx \\ &= -2\alpha \int_0^1 v(x)^2\,dx \le 0\end{aligned}$$

and therefore B_ν is dissipative. Moreover, given $(f,g) \in H$, the equations

$$\begin{cases} u(x) - v(x) = f(x) \\ -(a_\nu(x)u'(x))' + (2\alpha+1)v(x) + u(x) = g(x) \end{cases}$$

have an unique solution $(u,v) \in D(C_\nu)$, which shows that $R(I - B_\nu) = H$. By the Lumer-Phillips Theorem, B_ν is the generator of a strongly continuous semigroup of contractions on H. Since $C_\nu = B_\nu + N$, where $N(u,v) = (0,-u)$ is a bounded linear operator on H, it follows that C_ν is also the generator of a strongly continuous semigroup $\{e^{C_\nu t}, t \ge 0\}$ in H.

We now study the spectrum of C_ν and prove exponential decay in a subspace of H.

Lemma 3.3 *Let $0 = \lambda_0(\nu) < \lambda_1(\nu) < \ldots \to \infty$ be the sequence of the eigenvalues and $\phi_{0\nu}(x) \equiv 1$, $\phi_{1\nu}, \ldots$ be the corresponding eigenfunctions of*

$$\begin{cases} -(a_\nu(x)\phi_{j\nu}(x))' = \lambda_j(\nu)\phi_{j\nu}(x), 0 < x < 1, \\ \phi'_{j\nu}(x) = 0,\ \textit{for } x = 0,1 \end{cases}$$

(that is: the eigenvalues and eigenfunctions of A_ν).

As $\nu \to 0+$, we have $\lambda_1(\nu) \to \frac{a_1 l_1}{2l_1 x_1(1-x_1)}$ and $\lambda_n(\nu) = O(\frac{1}{\nu})$, for all $n \ge 2$. Futhermore, $\|\phi_1(\nu)\|_{L^\infty(0,1)}$ is bounded and we have

$$\phi_{1\nu}(x) \to \begin{cases} -\sqrt{\frac{x_1}{1-x_1}} + O(\nu^{\frac{1}{2}}),\ \textit{if } x \in [0, x_1 - \nu\ell'_1] \\ -\sqrt{\frac{1-x_1}{x_1}} + O(\nu^{\frac{q}{2}}),\ \textit{if } x \in [x_1 - \nu\ell'_1, x_1 - \nu\ell_1] \\ O(1),\ \textit{if } x \in x_1 - \nu\ell_1, x_1 + \nu\ell_1] \\ \sqrt{\frac{x_1}{1-x_1}} + O(\nu^{\frac{q}{2}}),\ \textit{if } x \in [x_1 - \nu\ell'_1, x_1 - \nu\ell_1] \\ \sqrt{\frac{x_1}{1-x_1}} + O(\nu^{\frac{1}{2}}),\ \textit{if } x \in [x_1 + \nu\ell'_1, 1], \end{cases} \tag{3.5}$$

as $\nu \to 0+$.

The spectrum of C_ν is given by $\Sigma = \{\mu_k^\pm \ , \ k = 0, 1, 2, \cdots, \}$, where $\mu_k^\pm(\nu) = -\alpha \pm \sqrt{\alpha^2 - \lambda_k(\nu)}$.

If ν is small enough, we have four real eigenvalues $\mu_0^\pm(\nu), \mu_1^\pm(\nu)$, with corresponding normalized eigenfunctions $\psi_k^\pm = \frac{1}{\sqrt{1-2\alpha\mu_k^\pm(\nu)}}(\phi_k, \mu_k^\pm(\nu)\phi_k), \ k = 0, 1.$

Corresponding to each pair of complex eigenvalues $\mu_k^\pm(\nu), \ k \geq 2$ there is an orthonormal basis in the two-dimensional generalized eigenspace $\psi_k^1 = (\frac{1}{\sqrt{\lambda_k+1}}\phi_k, 0) \ , \ \psi_k^2 = (0, \phi_k)$.

We then have the basis in H $\left\{\psi_0^+, \psi_1^+, \psi_0^-, \psi_1^-, \psi_{k\geq 2}^1, \psi_{k\geq 2}^2\right\}$.

We observe that $\langle \psi_1^+, \psi_1^- \rangle = \frac{1+2\lambda_1}{\sqrt{1+4\alpha^2(1+\lambda_1)}}$ and $\langle \psi_i^\pm, \psi_j^{-\pm} \rangle = \delta_{ij}$ in any other case.

The matrix of the operator C_ν with respect to this basis is given by

$$\begin{bmatrix} 0 & & & & & & \\ & \mu_1^+ & & & & & \\ & & -2\alpha & & & & \\ & & & \mu_1^- & & & \\ & & & & C_2 & & \\ & & & & & C_3 & \\ & & & & & & \cdots \end{bmatrix}$$

where

$$C_n = \begin{bmatrix} 0 & \sqrt{\lambda_n + 1} \\ \frac{-\lambda}{\sqrt{\lambda_n+1}} & -2\alpha \end{bmatrix}$$

It is convenient at this point to introduce a new inner product in H. To this end we observe that any $z \in H$ can be written (in a unique way) as

$$z = x_0^+\psi_0^+ + x_0^-\psi_0^- + x_1^+\psi_1^+ + x_1^-\psi_1^- + \sum_{n=2}^{\infty} x_n^1\psi_n^1 + x_n^2\psi_n^2$$

and define

$$\langle\langle z, \tilde{z} \rangle\rangle = x_0^+\tilde{x}_0^+ + x_0^-\tilde{x}_0^- + x_1^+\tilde{x}_1^+ + x_1^-\tilde{x}_1^- + \sum_{n=2}^{\infty} x_n^1\tilde{x}_n^1 + x_n^2\tilde{x}_n^2$$

For later purposes, we observe here that

$$x_0^+ \ = \ \frac{\left| \begin{matrix} < z, \psi_0^+ > & < \psi_0^-, \psi_0^+ > \\ < z, \psi_0^- > & < \psi_0^-, \psi_0^- > \end{matrix} \right|}{\left| \begin{matrix} < \psi_0^+, \psi_0^+ > & < \psi_0^-, \psi_0^+ > \\ < \psi_0^+, \psi_0^- > & < \psi_0^-, \psi_0^- > \end{matrix} \right|}$$

$$= \frac{1+4\alpha^2}{4\alpha^2} < z, \psi_0^+ > - \frac{\sqrt{1+4\alpha^2}}{4\alpha^2} < z, \psi_0^- > \tag{3.6}$$

$$x_1^+ = \frac{\left| \begin{array}{cc} < z, \psi_1^+ > & < \psi_1^-, \psi_1^+ > \\ < z, \psi_1^- > & < \psi_1^-, \psi_1^- > \end{array} \right|}{\left| \begin{array}{cc} < \psi_1^+, \psi_1^+ > & < \psi_1^-, \psi_1^+ > \\ < \psi_1^+, \psi_1^- > & < \psi_1^-, \psi_1^- > \end{array} \right|}$$

$$= \frac{1+4\alpha^2(1+\lambda_1)}{4(\alpha^2-\lambda_1)(1+\lambda_1)} < z, \psi_1^+ > - \frac{\left(\sqrt{1+4\alpha^2(1+\lambda_1)}\right)(1+2\lambda_1)}{4(\alpha^2-\lambda_1)(1+\lambda_1)} < z, \psi_1^- \tag{3.7}$$

Now, denoting the new norm by $|||\cdot|||$, we have

$$\begin{aligned} ||z||^2 &= {x_0^+}^2 + {x_0^-}^2 + {x_1^+}^2 + {x_1^-}^2 + \sum_{n=2}^{\infty} {x_n^1}^2 + {x_n^2}^2 + x_0^+ x_0^- \langle \psi_0^+, \psi_0^- \rangle + x_1^+ x_1^- \langle \psi_1^+, \psi_1^- \rangle \\ &\leq |||z|||^2 + 1/2 \left({x_0^+}^2 + {x_0^-}^2\right) |\langle \psi_0^+, \psi_0^- \rangle| + 1/2 \left({x_1^+}^2 + {x_1^-}^2\right) |\langle \psi_1^+, \psi_1^- \rangle| \end{aligned}$$

and

$$||z||^2 \geq |||z|||^2 - 1/2 \left({x_0^+}^2 + {x_0^-}^2\right) |\langle \psi_0^+, \psi_0^- \rangle| - 1/2 \left({x_1^+}^2 + {x_1^-}^2\right) |\langle \psi_1^+, \psi_1^- \rangle|$$

If $\alpha^2 \geq \lambda_1$, then $|\langle \psi_0^+, \psi_0^- \rangle|$, $|\langle \psi_1^+, \psi_1^- | \leq 1$ and it follows that

$$1/2|||z|||^2 \leq ||z||^2 \leq 3/2|||z|||^2$$

Furthermore, the basis defined above is, by construction, orthonormal in the new inner product, which we adopt from now on. We also denote it simply by $\langle \cdot \rangle$.

We now write $z = x_0^+ \psi_0^+ + x_1^+ \psi_1^+ + w$ where $w \in W = [\psi_0^+, \psi_1^+]^\perp$.

Lemma 3.4 *Let $\tilde{C} = C_{|W}$ be the restrition of C to the subspace W. Then*

$$||e^{t\tilde{C}} w|| \leq e^{-\alpha t} ||w||$$

for any $t > 0$, $w \in W$.

Proof It is enough to prove the inequality in each of the invariant orthogonal eigenspaces $X_0^- = [\psi_0^-], X_1^- = [\psi_1^+]$ and $X_n = [{\psi_n}^1, {\psi_n}^2]$, $n = 2, 3, \cdots$. We treat only the two dimensional eigenspaces X_n, since the proof is easier in the one dimensional cases.

If $w \in X_n$, $w = a_n{\psi_n}^1 + b_n{\psi_n}^2$ then $||w|| = |(a,b)|$ where $|\cdot|$ denotes euclidean norm. Thus $||e^{\tilde{C}t}w|| = |e^{C_n t}(a,b)|$.
Let $P = \begin{bmatrix} \sqrt{1+\lambda_n} & 0 \\ -\alpha & \sqrt{\lambda_n - \alpha^2} \end{bmatrix}$. Then

$$P^{-1}C_nP = \begin{bmatrix} -\alpha & \sqrt{\lambda_n - \alpha^2} \\ -\sqrt{\lambda_n - \alpha^2} & -\alpha^2 \end{bmatrix} = B_n$$

It follows that

$$\begin{aligned} |e^{C_n t}(a,b)| &\leq |e^{B_n t}(a,b)| \\ &= |e^{-\alpha t}\begin{bmatrix} \cos \alpha t & \sin(\sqrt{\lambda_n - \alpha^2}t) \\ -\sin(\sqrt{\lambda_n - \alpha^2}t) & \cos \alpha t \end{bmatrix}(a,b)| \\ &\leq e^{-\alpha t}|(a,b)| \end{aligned}$$

which proves the claim.

4 Well-posedness and existence of attractors

In this section we discuss the well-posedness of problem (**??**), (**??**) and prove that it generates a dynamical system in $H^1(0,1) \times L^2(0,1)$. Although these results are well known we include them here for the sake of completeness. We also show that the problem has a global compact attractor which is bounded in the L^∞-norm by a constant independent of ν and α.

With $z = (u,v) \in H$ and $G : H \to H$ defined by $G(z)(x) = (0, f(u(x)))$, $0 < x < 1$,equation (**??**) can be written as an evolution equation

$$\dot{z}(t) = C_\nu z(t) + G(z), \tag{4.1}$$

or in a "matrix notation" that will be convenient later:

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} 0 & I \\ -A_\nu & -2\alpha \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} 0 \\ f^e(u) \end{pmatrix}$$

Our first goal is to show that the initial value problem for (**??**) has a global unique solution. As usual, solutions of (**??**) are defined as continuous solutions of the integral equation

$$z(t) = e^{C_\nu t}z(0) + \int_0^t e^{C_\nu(t-s)}G(z(s))\, ds. \tag{4.2}$$

We denote by $\mathcal{C}^{1,1}$ the set of differentiable functions with Lipschitz continuos derivatives.

Lemma 4.1 *If $f : \mathbb{R} \to \mathbb{R}$ is $C^{1,1}$, then $G : H \to H$ is $C^{1,1}$ and compact. Moreover, G and DG maps bounded sets into bounded sets.*

Proof. (PROVA DA PRIMEIRA PARTE?) We prove first that G is compact. Observe that $G = \Psi \circ F \circ \Phi$ where $\Phi : H \to H^1, \quad \Psi : L^2 \to H, \quad F : H^1 \to L^2$ are given by$\Phi(u, v) = u, \quad \Psi(w) = (0, w) \quad F(u)(x) = f(u(x))$.

Since Φ and Ψ are clearly continuous we only have to prove that F is compact. Also since by Sobolev's imbeding theorem, the imersion $H^1 \hookrightarrow L^2$ is compact it is enough to prove that F, considered as function from H^1 into H^1 is bounded. Suppose then, that $||u||_{H^1} \leq C$, where C is a real positive constant. Since $H^1 \hookrightarrow L^\infty$ with continuous imersion, it follows that $||u||_\infty \leq K$, where K is a positive constant depending only ond C. Therefore, there exists a constant M (depending only on C), such that $sup\,\{f(u(x)), f'(u(x))\}$, where u varies in the set $\{u \in H^1, ||u||_{H^1} \leq C\}$. Thus

$$\begin{aligned}
||F(u)||_{H^1} &= \left(\int_0^1 (f(u(x)))^2 + (f'(u(x)))^2 \, d\, x\right)^{1/2} \\
&\leq \left(M^2 + M^2 \int_0^1 (u'(x))^2\right)^{1/2} \\
&\leq \left(M^2 + M^2 ||u||_{H^1}^2\right)^{1/2} \\
&\leq M\left(1 + ||u||_{H^1}^2\right)^{1/2} \\
&\leq M\left(1 + C^2\right)^{1/2}
\end{aligned}$$

So G is compact (and therefore also bounded).

Now, if $h = (\xi, \eta) \in H$ $(G'(z) \cdot h)(x) = (0, f'(u(x)) \cdot \xi(x)$

Reasoning as before, all we have to prove is that $F' : H^1 \longrightarrow \mathcal{L}(H^1, L^2)$ is bounded.

Take $h \in H^1$. Then $||F'(u)(h)||_{L^2}^2 = \int_0^1 ||f'(u(x)) \cdot h(x)||^2 \, d\, x \leq M \int_0^1 ||h(x)||^2 \, d\, x \leq M||h||_{H^1}$ which proves proves the claim.

By using standard arguments and the Contraction Mapping Principle (see [**?**], [**?**]) we can prove the following

Theorem 4.2 *For any $z_0 \in H$, there exists $0 < \beta \leq \infty$ and a unique solution $z(\cdot, z_0)$ of (**??**) defined on $[0, \beta)$ satifying $z(0) = z_0$. If $\beta < \infty$, then $\limsup_{t \to \beta-} \|z(t)\| = \infty$. If $z_0 \in D(C_\nu)$, then $t \in (0, \beta) \mapsto z(t, z_0) \in H$ is C^1, $t \in [0, \beta) \mapsto z(t, z_0) \in D(C_\nu)$ is continuous and z is a strict solution of* (**??**).

Next we show that all solutions are bounded and therefore defined on $[0, \infty)$. In fact, we show that there exists a bounded set in H which attracts any bounded set of H under the flow generated by (**??**), (**??**). To this end, we consider the Liapunov function

$$V(u,v) = \int_0^1 \left[\frac{v(x)^2}{2} + \frac{a_\nu(x) u'(x)^2}{2} \right] - F(u)$$

where $F(u) = \int_0^u f(s)\, ds$.

Lemma 4.3 *If B is a bounded subset of H, then $\cup_{t \geq 0} T(t)B$ is also a bounded subset of H. In particular, any orbit of (**??**) is bounded.*

Proof. Hypothesis (**??**) implies that there exist positive constants k_1 and k_2 such that $uf(u) \leq \frac{-r}{2} u^2 + k_1$ and $F(u) \leq \frac{-r}{4} u^2 + k_2$, for all $u \in \mathbb{R}$. Therefore,

$$\begin{aligned} V(u,v) &\geq \int_0^1 \left\{ \frac{v(x)^2}{2} + \frac{r}{4} u(x)^2 + \frac{1}{2} a_\nu(x) u'(x)^2 - k_2 \right\} dx \\ &\geq C_1 ||(u,v)||^2 - k_2 \end{aligned} \tag{4.3}$$

where $C_1 = \min\{\frac{1}{2}, \frac{r}{4}\}$. (Observe that C_1 does not depend on ν or α.

Now, if $z(t) = (u(\cdot,t), v(\cdot,t))$ is a solution of (**??**) with initial value in $D(C_\nu)$, then

$$\begin{aligned} \frac{d}{dt} V(u(\cdot,t), v(\cdot,t)) &= \int_0^1 \{ v v_t + a_\nu(x) u_x u_{xt} - f(u) u_t \}\, dx \\ &= \int_0^1 -2\alpha v^2\, dx \end{aligned} \tag{4.4}$$

Therefore $V(u(t), v(t))$ decreases if $(u(t), v(t)$ is a solution with initial value in $D(C_\nu)$ and then, by continuity, this is also true for any solution.

Suppose $B \subset H$ is bounded, say $B \subset B(0,r)$. The there exists a constant $C_2 = C_2(B)$ such that $\|u\|_\infty \leq C_2$ if $(u,v) \in B$ (and this constant can be chosen independently of α and ν.

Therefore, there exists $C_3 = C_3(B)$ such that $|f(s)| \leq C_3$ if $0 \leq s \leq C_3$. Thus $|F(u(x))| \leq \int_0^{u(x)} |f(s)|\, ds \leq C_2 C_3$ and $\int_0^1 |F(u(x))|\, dx \leq C_2 C_3$, so

$$|V(u,v)| \leq \frac{1}{2} \|(u,v)\|_H^2 + C_2 C_3 \leq \frac{1}{2} R^2 + C_2 C_3$$

We conclude that, for $t \geq 0$ and $(u,v) \in B$

$$\|T(t)(u,v)\| \leq V(T(t)(u,v)) + k_2 \leq V(u,v) + k_2 \leq \frac{1}{2} R^2 + C_2 C_3 + k_2$$

(Observe that these constants do not depend on ν or α).

Lemma 4.4 *The semigroup $T(t)$ is asymptoticaly smooth.*

From the variation of constants formula and lemma **??** it follows that

$$T(t)z = e^{C_\nu t}z + U(t)z$$

where $U(t)$ is compact.

Writing $z = x_0^+\psi_0^+ + x_1^+\psi_1^+ + w$, it follows from the representation of C_ν given in section **??** that

$$T(t)z = e^{C_\nu t}w + {x_1}^+ e^{\mu_1^+ t}\psi_1^+ + x_0^+\psi_0^+ + U(t)z$$

Writing $S(t) = e^{C_\nu t}w + {x_1}^+ e^{\mu_1^+ t}\psi_1^+$ and $V(t) = x_0^+\psi_0^+ + U(t)z$, we have $T(t) = S(t) + V(t)$, with $\|S(t)z\| \leq e^{\mu_1^+ t}z\|$ by lemma **??** and $V(t)$ compact.

Using lemma 3.2.6 of [**?**] the result follows immediately.

For convenience we trancribe here the definition of a gradient system as given in [**?**].

Definition 4.1 *A strongly continous $\mathcal{C}^r$-semigroup $T(t) : X \to X$, $t \geq 0$, $r \geq 1$ is said to be a* gradient system *if*

1. *Each bounded orbit is precompact.*
2. *There exists a Lyapunov function for $T(t)$; that is, there is a continuos function $V : X \to \mathbb{R}$ with the property that*
   - *(a) V is bounded below.*
   - *(b) $V(x) \to \infty$ as $t \to \infty$.*
   - *(c) $V(T(t)x)$ is noincreasing in t for each $x \in X$.*
   - *(d) If x is such that $T(t)x$ is defined for $t \in \mathbb{R}$ and $V(T(t)x) = V(x)$ for $t \in \mathbb{R}$, then x is and equilibrium point.*

Lemma 4.5 *T(t) is a gradient system*

Proof. **??**. follows from lemmas **??**, **??** and the variation of constants formula. **??**), **??**), **??**) follow from **??** and **??** (in the proof of lemma **??**). Finally, suppose that $V(T(t)z) = V(z)$ for $t \in \mathbb{R}$, where $z = (u(t), v(t))$. Then, by **??**, $v(t,x) \equiv 0$ and so $u(t,x) = u(0,x) = \varphi(x)$ and satisfies

$$(a\varphi(x))_x + f(\varphi(x)) = 0\ ,\ \varphi_x(0) = \varphi_x(1) = 0$$

that is $z = (\varphi, 0)$ is an equilibrium. This proves **??**.

Lemma 4.6 *Let E be the set of equilibrium points of $T(t)$. Then, there exists a constant C, independent of ν and α such that $\|\varphi\| \leq C$, for all $(\varphi, 0) \in E$.*

Proof. If $(\varphi, 0) \in E$ then $(a_\nu(x)\varphi(x))_x + f(\varphi(x)) = 0$, $\varphi_x(0) = \varphi_x(1) = 0$. Multiplying by $\varphi(x)$ and integrating, we obtain

$$\int_0^1 (a_\nu(x)\varphi(x))\varphi_x \, dx = \int_0^1 f(\varphi(x)) \, dx$$

From **??**, it follows that

$$\int_0^1 (a_\nu(x)\varphi_x(x))\varphi_x(x) \leq \int_0^1 -\frac{r}{2}u^2(x) + k_1 \, dx$$

and so

$$\begin{aligned} \|\varphi\|^2 &= \int_0^1 (a_\nu(x)\varphi_x(x))\varphi_x(x)) + u^2(x) \, dx \\ &\leq \max\{1, \frac{2}{r}\} \int_0^1 k_1 \, dx \end{aligned}$$

which gives the result.

Theorem 4.7 *The problem (**??**) has a global compact attractor $\mathcal{A}$ for any $\nu > 0$ and $\alpha > 0$, and $\mathcal{A} = W^u(E)$. If each element of E is hyperbolic then $\mathcal{A} = \bigcup_{x \in E} W^u(x)$. Furthermore there exists a constant C independent of ν and α such that $\|u\| \leq C$ for any $(u, v) \in \mathcal{A}$.*
Proof. *The existence follows from theorem 3.8.5 of [**?**]. For the boundeness we use lemma **??**, the decreasing property of V and the imbbeding given by lemma **??**.*

5 Invariant Manifold and Asymptotic Behavior

Using the notation of section **??** we write $z = x_0^+\psi_0^+ + x_1^+\psi_1^+ + w$ for $z \in H$.

Then equation **??** can be decomposed as

$$\begin{cases} \dot{x}_0^+ &= P_0(x_0^+, x_1^+, w) \\ \dot{x}_1^+ &= \mu_1^+ x_1^+ + P_1(x_0^+, x_1^+, w) \\ \dot{w} &= C_3 w + Q(x_0^+, x_1^+, w) \end{cases}$$

where

$$\begin{aligned} C_3 &= \begin{bmatrix} 0 & I \\ -A_{|W} & -2\alpha \end{bmatrix} \\ P_0(x_0^+, x_1^+, w) &= < G(z), \psi_0^+ > \\ P_1(x_0^+, x_1^+, w) &= < G(z), \psi_1^+ > \\ Q(x_0^+, x_1^+, w) &= G(z) - P_0(x_0^+, x_1^+, w)\psi_0^+ - P_0(x_0^+, x_1^+, w)\psi_1^+ \end{aligned}$$

Using equations **??** e **??** (with $z = (0, f(u))$), we obtain (after some routine computation)

$$P_0(x_0^+, x_1^+, w) = \frac{1}{2\alpha}\int_0^1 f(z)\phi_0 \tag{5.5}$$

$$P_1(x_0^+, x_1^+, w) = \frac{\sqrt{1-2\alpha\mu_1^+}}{\mu_1^+ - \mu_1^-}\int_0^1 f(z)\phi_1 \tag{5.6}$$

We now proceed formally in order to obtain a set of O.D.Es, which we expect to describe the asymptotic behavior of **??** if α is big and ν goes to infinity.

If α is big we expect, from lemma **??**, the w-component to die out very fast. Taking also into account the behavior of the eigenvalues and eigenfunctions of A, given by lemma **??** as $\nu \to 0$, we obtain

$$\begin{aligned} P_0(x_0^+, x_1^+, w) &= \frac{1}{2\alpha}\int_0^1 f(x_0^+\phi_0 + x_1^+\frac{1}{\sqrt{1-2\alpha\mu_1^+}}\phi_1 + w) \\ &\sim \frac{1}{2\alpha}\left(\int_0^{x_1} f(x_0^+ + x_1^+\frac{k_1}{\gamma}) + \int_{x_1}^1 f(x_0^+ + x_1^+\frac{k_2}{\gamma})\right) \\ &= \frac{1}{2\alpha}\left(x_1 f(x_0^+ + x_1^+\frac{k_1}{\gamma}) + (1-x_1)f(x_0^+ + x_1^+\frac{k_2}{\gamma})\right) \end{aligned} \tag{5.7}$$

$$\begin{aligned} P_1(x_0^+, x_1^+, w) &= \frac{\sqrt{1-2\alpha\mu_1^+}}{\mu_1^+ - \mu_1^-}\int_0^1 f(x_0^+\phi_0 + x_1^+\frac{1}{\sqrt{1-2\alpha\mu_1^+}}\phi_1 + w)\phi_1 \\ &\sim \frac{\sqrt{\gamma}}{2\sqrt{\alpha^2-\beta}}\left(\int_0^{x_1} f(x_0^+ - x_1^+\frac{k_1}{\gamma})k_1 + \int_{x_1}^1 f(x_0^+ + x_1^+\frac{k_2}{\gamma})\right)k_2 \\ &= \frac{\gamma}{2\sqrt{\alpha^2-\beta}}\left(-x_1k_1 f(x_0^+ - x_1^+\frac{k_1}{\gamma}) + (1-x_1)k_2 f(x_0^+ + x_1^+\frac{k_2}{\gamma})\right) \end{aligned} \tag{5.8}$$

where

$$\begin{aligned} \beta &= \lim_{\mu\to 0}\lambda_1 = \frac{2x_1 l_1}{x_1(1-x_1)} \\ \gamma &= \lim_{\nu\to 0}\sqrt{1-2\alpha\mu_1^+} = 1 + \frac{2\beta}{1+\sqrt{1-\frac{\beta}{\alpha}}} \end{aligned}$$

After the change of variables

$$\begin{array}{rcl} x_0^+ & = & x_1 z_1 + (1-x_1) z_2 \\ x_1^+ & = & \gamma\left(-k_1 x_1 z_1 + k_2 (1-x_1) z_2\right) \end{array} \tag{5.9}$$

equations **??**, **??** become

$$\begin{array}{rcl} \dot{z_1} & = & \left(\frac{x_1}{2\alpha} + \frac{1-x_1}{2\sqrt{\alpha^2-\beta}}\right) f(z_1) + \left(\frac{1-x_1}{2\alpha} - \frac{1-x_1}{2\sqrt{\alpha^2-\beta}}\right) f(z_2) \\ & + & (\sqrt{\alpha^2-\beta}-\alpha)(1-x_1)(z_1-z_2) \\ & = & \frac{1}{2\alpha}\left(f(z_1) + (1-x_1)\beta(z_2-z_1)\right) \\ & + & \frac{1}{2\alpha^3}\left(\frac{\beta^2}{2-\frac{\beta}{\alpha^2}+2\sqrt{1-\frac{\beta}{\alpha^2}}}(z_2-z_1) - (1-x_1)\frac{\beta}{1-\frac{\beta}{\alpha^2}+\sqrt{1-\frac{\beta}{\alpha^2}}(f(z_2)-f(z_1))}\right) \\ \dot{z_2} & = & \left(\frac{x_1}{2\alpha} - \frac{x_1}{2\sqrt{\alpha^2-\beta}}\right) f(z_1) + \left(\frac{1-x_1}{2\alpha} + \frac{x_1}{2\sqrt{\alpha^2-\beta}}\right) f(z_2) \\ & + & (\sqrt{\alpha^2-\beta}-\alpha)(x_1)(z_2-z_1) \\ & = & \frac{1}{2\alpha}\left(f(z_2) - x_1\beta(z_2-z_1)\right) \\ & + & \frac{1}{2\alpha^3}\left(\frac{\beta^2}{2-\frac{\beta}{\alpha^2}+2\sqrt{1-\frac{\beta}{\alpha^2}}}(z_2-z_1) + x_1\frac{\beta}{1-\frac{\beta}{\alpha^2}+\sqrt{1-\frac{\beta}{\alpha^2}}(f(z_2)-f(z_1))}\right) \end{array}$$

Reescaling the time by 2α, we obtain

$$\begin{array}{rcl} \dot{z_1} & = & \left(f(z_1) + \frac{a_1}{2l_1 x_1}(z_2-z_1)\right) + \frac{1}{2\alpha^2}R_1(\alpha) \\ \dot{z_2} & = & \left(f(z_2) - \frac{a_1}{2l_1(1-x_1)}(z_2-z_1)\right) + \frac{1}{2\alpha^2}R_1(\alpha) \end{array}$$

where $R_1(\alpha), R_2(\alpha) = O(1)$ as $\alpha \to \infty$.

As α grows, these equations approach the system

$$\left\{ \begin{array}{rcl} \dot{z_1} & = & f(z_1) + \frac{a_1}{2l_1 x_1}(z_2-z_1) \\ \dot{z_2} & = & f(z_2) - \frac{a_1}{2l_1(1-x_1)}(z_2-z_1) \end{array} \right. \tag{5.10}$$

which determines the asymptotic behavior in the parabolic case (see [**?**], [FUSCO]). This is in agreement with the results of [?], showing the attractor of the hyperbolic equation $hyperbolic - with - \epsilon$ approachs the attractor of the $parabolic$.

We now want to proceed in showing that these equations indeed describe the asympotic behavior of **??**. To this end we first modify the function G outside a ball of H containing the attractor for all ν and α, in order to obtain a function that is globally bounded in $\mathcal{C}^{1,1}(H,H)$, that is, the space of functions in H whose derivative satisfies a Lipschitz condition. The existence of a ball B of H with the required property is granted by theorem **??**. Furthermore, from lemma **??**, there exists a constant C such that $\{u(x)|z=(u,v)\in B\}\subset[-C,C]$ We then modify f outside $[-C,C]$ in such a way that f becomes bounded in $\mathcal{C}^{1,1}(\mathrm{I\!R},\mathrm{I\!R})$, It is then easily verified (needs a proof?) that G satisfies the required properties.

We will now prove that the modified flow admits a (global) exponentially attracting invariant manyfold. This will, of course, be only a *local* invariant manyfold for the original system. But, since it contains the *global* attractor of the original equation, the asymptotic behavior is determined by the flow in it.

The following result can be proved following the lines of Henry [**?**] (Thms. 6.1.2,6.1.4,6.1.5,6.1.7). A somewhat different proof can be found in [Chow].

Theorem 5.1 *Let X,Y be Banach spaces with $dim Y<\infty$ and assume that A is the generator of a $\mathcal{C}^0$ semigroup. Suppose $F: X\times Y\to X$ and $G: X\times Y\to Y$ are bounded in $\mathcal{C}^{1,1}(X\times Y, X\times Y)$.*

Consider the system

$$\begin{cases} \dot{x} &= Ax+F(x,y) \\ \dot{y} &= G(x,y) \end{cases} \tag{5.11}$$

Assume that

1. *$||F(x,y)-F(x',y')||\le\lambda\left(||x-x'||+||y-y'||\right)$, $\quad ||F(x,y)||\le N$, for any $(x,y)\in X\times Y$.*

2. *$||e^{-At}||\le M||e^{-\alpha t}||$ for $t>0$.*

3. *$||G(x,y)-G(x',y')||\le M_2\left(||x-x'||+||y-y'||\right)$ for any $(x,y)\in X\times Y$.*

Suppose that, for some positive constants Δ and D

1. *$\frac{N}{\alpha}<D$.*

2. *$\theta=\frac{\lambda}{\alpha-(1+\Delta)M_2}<\frac{\Delta}{1+\Delta}$.*

3. *$\frac{2\lambda}{\alpha-2(1+\Delta)M_2}<1$.*

Then there exists a (global) exponentially attracting invariant manifold

$$S = \{(x, y) \mid x = \sigma(y), y \in Y\}$$

which is bounded in $\mathcal{C}^{1,1}(X \times Y, X \times Y)$*, and satisfies*
$||\sigma(y)|| \leq D$*,* $||\sigma(y_1) - \sigma(y_2)|| \leq \Delta||y_1 - y_2||$.

Corollary 5.1 *Suppose* f *belongs to* $\mathcal{C}^{1,1}(\mathbb{R}, \mathbb{R})$ *with* $\sup\{f(x)|x \in \mathbb{R}\} = M$*,* $\sup\{f_x(x)|x \in \mathbb{R}\} = L$*. Assume that*

a) $\sqrt{\lambda_2(\nu)} > \alpha > 6L$.

Then, there exists a (global) exponentially attracting invariant manifold

$$S = \left\{(x_0^+, x_1^+, w) \mid w = \sigma(x_0^+, x_1^+), \in \mathbb{R}^2\right\}$$

*for system [**??**].*
The flow on S *is given by* $u(t, x) = x_0^+\phi_0(x) + x_1^+\phi_1(x) + \sigma(x_0^+, x_1^+)$ *where* (x_0^+, x_1^+) *is the solution of*

$$\left\{\begin{array}{lcl} \dot{x}_0^+ & = & P_0(x_0^+, x_1^+, \sigma(x_0^+, x_1^+)) \\ \dot{x}_1^+ & = & \mu_1^+ x_1^+ + P_1(x_0^+, x_1^+, \sigma(x_0^+, x_1^+)) \end{array}\right. \tag{5.12}$$

As α *and* $\nu \to 0$*, (with* $\lambda_2^{\frac{1}{2}} > \alpha$*)* $\sigma \to 0$ *in* $\mathcal{C}^1(\mathbb{R}^2, W)$*.*

Proof. If f satisfies the conditions above, it is easy to prove that P_0, P_1 and Q belong to $\mathcal{C}^{1,1}(H, H)$ with the same bounds L and M. The first part follows from the theorem **??** with $\Delta = 1$ and D any number bigger than $\frac{M}{\alpha}$. For the second part we observe that Δ and D can be taken as small as wished by taking α big enough.

Theorem 5.2 *Suppose the system* **??** *is structurally stable. Then, for* α *sufficiently big and* ν *sufficiently small, with* $\sqrt{\lambda_2(\nu)} > \alpha$ *the flow on the invariant manifold given by* **??** *is topologically equivalent to* **??***.*

Proof. For each α and ν satisfying the conditions of corolary **??** consider the flow in the two-dimensional subspace generated by ϕ_0 and ϕ_1, given by $v(t, x) = x_0^+\phi_0(x) + x_1^+\phi_1(x)$, where

$$\left\{\begin{array}{lcl} \dot{x}_0^+ & = & P_0(^+, x_1^+, \sigma(x_0^+, x_1^+)) \\ \dot{x}_1^+ & = & \mu_1^+ x_1^+ + P_1(x_0^+, x_1^+, \sigma(x_0^+, x_1^+)) \end{array}\right. \tag{5.13}$$

Since the application $v \to u = v + \sigma u$ is a conjugation betwen orbits of this flow and orbits of **??**, we only have to prove that system **??** is topologically conjugate to **??**.

Now, reescaling the time by $\frac{1}{2\alpha}$, and changing back the variables to x_0^+, x_1^+ by using **??**, we see that what remains to be proved is that the vector fields

$$X_{(\alpha,\nu)}(x_0^+, x_1^+) = \left(\tilde{P}_0(\alpha,\nu), \tilde{P}_1(\alpha,\nu)\right)$$

and

$$X_{(\infty,0)}(x_0^+, x_1^+) = \left(\tilde{P}_0(\infty,0), \tilde{P}_1(\infty,0)\right)$$

where

$$\begin{aligned}
\tilde{P}_0(\alpha,\nu) &= \int_0^1 f(x_0^+\phi_0(x) + x_1^+\phi_1(x) + \sigma(x_0^+, x_1^+))\phi_0 \\
\tilde{P}_1(\alpha,\nu) &= 2\alpha\frac{\sqrt{1-2\alpha\mu_1^+}}{\mu_1^+ - \mu_1^-}\int_0^1 f(x_0^+\phi_0(x) + x_1^+\phi_1(x) + \sigma(x_0^+, x_1^+))\phi_1 \\
\tilde{P}_0(\infty,0) &= \left(x_1 f(x_0^+ + x_1^+\frac{k_1}{\gamma}) + (1-x_1)f(x_0^+ + x_1^+\frac{k_2}{\gamma})\right) \\
\tilde{P}_1(\infty,0) &= 2\alpha\frac{\gamma}{2\sqrt{\alpha^2-\beta}}\left(-x_1k_1f(x_0^+ - x_1^+\frac{k_1}{\gamma}) + (1-x_1)k_2f(x_0^+ + x_1^+\frac{k_2}{\gamma})\right)
\end{aligned}$$

are $\mathcal{C}^1$ close.
This follows easily from the fact that σ approaches 0 in the $\mathcal{C}^1$ topology and the asymptotic properties of the eigenvalues and eigenfunctions of C as $\nu \to 0$ and $\alpha \to \infty$.

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