Minimum degree conditions for cycles

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Abstract

In this note we discuss the lengths of cycles which are forced to exist in an \( n \)-vertex graph with minimum degree \( \delta \). We show that for any integer \( k \geq 2 \) there exists \( n_0 \) such that if \( n \geq n_0 \) and \( G \) is an \( n \)-vertex graph with \( \delta(G) = \delta \geq n/k \) then the following are true.

(i) \( G \) contains \( C_t \) for every even \( 4 \leq t \leq \left\lfloor \frac{n}{k} - 1 \right\rfloor \),
(ii) either \( G \) is in a known exceptional class or \( G \) contains \( C_t \) for every odd \( t \in \left\lfloor \frac{2n}{\delta} \right\rfloor - 1, \delta + 1 \), and
(iii) if \( G \) does not contain a cycle of every length from \( \left\lfloor \frac{2n}{\delta} \right\rfloor - 1 \) to \( \left\lceil \frac{n}{k} - 1 \right\rceil \) inclusive then \( G \) does contain \( C_t \) for every even \( 4 \leq t \leq 2\delta \).

This is an improvement on a theorem of Nikiforov and Schelp [7].

We recall that the circumference \( c(G) \) of a graph \( G \) is the length of the longest cycle in \( G \); we define also \( oc(G) \) and \( ec(G) \) to be the lengths of the longest odd and even cycles in \( G \).

Early results of Dirac [3] and Voss and Zuluaga [8] examined 2-connected graphs:

**Theorem 1.** (Dirac [3]) If \( G \) is a 2-connected graph with minimum degree \( \delta \) then \( c(G) \geq \min(|V(G)|, 2\delta) \).

**Theorem 2.** (Voss and Zuluaga [8]) If \( G \) is a 2-connected graph on at least \( 2\delta \) vertices with minimum degree at least \( \delta \) then \( ec(G) \geq 2\delta \); furthermore if \( G \) is not bipartite then also \( oc(G) \geq 2\delta - 1 \).

More recently various authors [1, 2, 4] removed the connectivity requirement, resulting in:

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Theorem 3. Let $k$ be any integer, and $G$ an $n$-vertex graph with minimum degree $\delta \geq \frac{n}{k}$. Then $c(G) \geq \left\lceil \frac{n}{k} \right\rceil$.

For illustration we prove the (very easy) corresponding result for paths.

Given a graph $G$ we define a block of $G$ to be a vertex-maximal 2-connected induced subgraph of $G$. We call a vertex whose removal increases the number of components of $G$ a cutvertex, and a block with only one cutvertex we call an endblock.

Theorem 4. If $G$ is an $n$-vertex graph with minimum degree $\delta$, then $G$ contains a path of length at least

$$p_{n,\delta} = \left\lceil \frac{n}{\left\lceil \frac{n}{\delta+1} \right\rceil} \right\rceil.$$ 

Proof. If $G$ contains a component with more than one block, let $X$ and $Y$ be distinct endblocks of this component with cutvertices $x$ and $y$ (it is possible that $x = y$). We greedily construct a path $P_x$ starting at $x$ and contained in $X$ with at least $\delta + 1$ vertices, and similarly a path $P_y$ from $y$ in $Y$, and join them with a path from $x$ to $y$ to obtain a path $P$ on at least $2\delta + 1 \geq p_{n,\delta}$ vertices.

If every component of $G$ is a block, by Dirac’s theorem there is certainly a path in $G$ of length $\min(2\delta, C)$ where $C$ is the size of the largest component of $G$. This quantity could only be smaller than $p_{n,\delta}$ if $C$ were smaller than $p_{n,\delta}$. But then by averaging $G$ would have to have a component with at most $\delta$ vertices, which is impossible. \hfill \Box

All these theorems are easily seen to be best possible; but one might wish to be able to find a cycle of some specified length. Nikiforov and Schelp [7] examined this problem, but their result was not best possible.

Naturally, one cannot expect to find odd cycles of any length in a graph $G$ with $\delta(G) \leq \frac{n}{2}$; $G$ may be bipartite. However, if one is told that $G$ is not bipartite, then a little more may be said.

We observe that there is a simple construction of non-bipartite graphs with large minimum degree and very restricted odd cycle lengths: given $\delta$, partition $[n]$ into sets $X_1, \ldots, X_r$ of size at least $2\delta$. On each set $X_i$ place a complete bipartite graph with parts of size at least $\delta$. Choose from each $X_i$ a vertex $v_i$. Now this graph satisfies the minimum degree conditions already and contains no odd cycles; we may place any $r$-vertex graph $H$ we desire on the vertices $\{v_1, \ldots, v_r\}$, and the odd cycle lengths of the resulting graph $G$ will be exactly those of $H$. We note that the vertices of $H$ are all cutvertices of $G$.

We define the set $B_k(n)$ to be those $n$-vertex graphs with minimum degree at least $n/k$ whose blocks are either bipartite or composed entirely of cutvertices (and therefore of size at most $k/2$). We now give our main theorem.

Theorem 5. Given an integer $k \geq 2$ there is $n_0$ such that whenever $n \geq n_0$ and $G$ is an $n$-vertex graph with minimum degree $\delta \geq n/k$, the following are true.

(i) $G$ contains $C_t$ for every even $4 \leq t \leq \left\lceil \frac{n}{k-1} \right\rceil$. 

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(ii) if $G$ is not in $\mathcal{B}_k(n)$ then $G$ contains $C_t$ for every odd $t \in [\lceil \frac{2n}{\delta} \rceil - 1, \delta + 1]$, and

(iii) if $G$ does not contain a cycle of every length from $\lceil \frac{2n}{\delta} \rceil - 1$ to $\left\lceil \frac{n}{k-1} \right\rceil$ inclusive then $G$ does contain $C_t$ for every even $4 \leq t \leq 2\delta$.

This theorem is a modification of that given by Nikiforov and Schelp [7], whose conditions (i) and (ii) were weaker (most importantly, they guaranteed even cycles only up to $\delta + 1$ vertices) and who gave no equivalent to condition (iii); our proof is also significantly shorter. Our theorem is best possible up to the value of $n_0$:

- We cannot hope to find longer even cycles in general since a graph consisting of a union of $k-1$ cliques on $\left\lceil \frac{n}{k-1} \right\rceil$ vertices, disjoint except for $(k-1) \left\lceil \frac{n}{k-1} \right\rceil - n < k-1$ vertices each of which is a cutvertex lying in two cliques, contains no longer even cycles and has minimum degree $\left\lceil \frac{n}{k-1} \right\rceil - 1$.

- We cannot expect to find shorter odd cycles since a blow-up of $C_{2\delta-1}$ with approximately equal parts contains none and has minimum degree at least $2\left\lfloor \frac{n}{2\delta-1} \right\rfloor$. We cannot expect to find longer odd cycles since a graph consisting of $K_{\delta+1}$ together with a complete balanced bipartite graph contains none (provided that $k \geq 4$; when $k = 2, 3$ we can replace $\delta + 1$ by $\left\lceil \frac{n}{k-1} \right\rceil$, the proof of which is trivial).

- Finally, we cannot expect even in the absence of odd cycles to find even cycles longer than $2\delta$ since the bipartite graph $K_{\delta,n-\delta}$ contains none.

Before giving the proof we give two well-known modifications of Dirac’s theorem.

**Lemma 6.** Given $d$ and $c \geq 1$, let $H$ be a graph on at most $2d - 9c$ vertices, with $\delta(H) \geq 7c$ and all but at most $2c$ vertices having degree at least $d$. Then $H$ contains a cycle of every length up to and including $|H|$.

*Proof.* We first consider the vertices with degree less than $d$. Since each such vertex has $7c$ neighbours, at least $5c$ of which have degree at least $d$, we can find two high-degree neighbours of each low-degree vertex creating a set of disjoint three-vertex paths. Now two high-degree vertices have at least $9c$ common neighbours, so we can form a path $P$ through all these three-vertex paths and at most $2c$ further high-degree vertices. We then take a path $P'$ of maximum length extending $P$; let $v$ be the right-hand endvertex of this path. There are at least $d - 8c$ neighbours of $v$ on $P' - P$, and hence if there were a (high-degree) vertex $w$ not on $P'$, we could find a neighbour of $w$ immediately right on $P'$ of a neighbour of $v$, and so extend the path. It follows that $P'$ covers $H$, and since both its endvertices have degree greater than $|H|/2$ there is a cycle $C$ covering $H$.

Now let $x$ and $y$ be successive high-degree vertices on $C$. Given any $3 \leq t < |H|$, for each neighbour $z$ of $x$ on $C$ there is an associated vertex $z'$ such that if $z'y$ is an edge of $H$, then there would be a cycle using parts of $C$, $xz$ and $yz'$ with exactly $t$ vertices. Since the vertices associated to distinct neighbours of $x$ are distinct and $2d > |H| + 1$, for at least one neighbour $z$ of $x$, $z'y$ is an edge of $H$, and so $H$ contains a cycle of length $t$. \qed
Lemma 7. Given a graph \( H \) with at most \( 2d - 8 \) vertices, minimum degree \( d \) and vertices \( x \neq y \), for each \( 3 \leq t \leq |H| - 2 \), there exists a \( t \)-vertex path \( P \) in \( H \) whose endvertices are \( x \) and \( y \).

The proof (whose detail we omit) is essentially the same as the previous; we find a cycle covering \( H \) on which \( x \) and \( y \) are either adjacent or at distance two, and shorten it (preserving the short \( x - y \) segment) to a desired length, yielding the desired path.

We also recall two theorems which will be useful.

Theorem 8. (Gould, Hazell and Scott [5]) For all \( c > 0 \) there exists \( K = K(c) \) such that if \( n > 45Kc^{-4} \) and \( G \) is an \( n \)-vertex graph with \( \delta(G) \geq cn \) then \( G \) contains \( C_t \) for every even \( t \in [4, ec(G) - K] \) and every odd \( t \in [K, oc(G) - K] \).

Theorem 9. (Häggkvist [6]) Given an integer \( l \geq 2 \), \( n \geq \binom{l+1}{2}(2l + 1)(3l - 1) \) and \( G \) an \( n \)-vertex graph with \( \delta(G) > \frac{2n}{2l+1} \), either \( G \) contains \( C_{2l-1} \) or it contains no odd cycle on more than \( l/2 \) vertices.

We now prove our main theorem.

Proof. Let \( G \) satisfy the conditions of the theorem. We suppose that \( n_0 \) is large enough that the inequalities in what follows hold.

Given a set \( X \) of vertices of \( G \) and a vertex \( v \), we say that \( v \) is dense to \( X \) if \( d_X(v) \geq 10k \), and sparse otherwise. Note that there are at most \( k \) cutvertices in \( G \), and that every cutvertex of \( G \) is dense to some block of \( G \).

First suppose that every block of \( G \) has less than \( 2\delta - 13k \) vertices. Let \( B_1, \ldots, B_r \) be the blocks of \( G \) which are not composed entirely of cutvertices, and for each \( i \) let \( b_i \) be the number of vertices in \( B_i \) that are not cutvertices of \( G \) sparse to \( B_i \). Then by Lemma 6, \( B_i \) contains a cycle of every length from 3 up to \( b_i \). Since every block \( B_i \) contains at least \( \delta + 1 > k + 10k^2 \) vertices, there is a vertex of \( B_i \) which is neither a cutvertex nor a neighbour of any sparse cutvertices: thus \( b_i \geq \delta + 1 \), and so \( r \leq k - 1 \). Since \( \sum_i b_i \geq n \) there is \( i \) such that \( b_i \geq \left\lceil \frac{n}{k-1} \right\rceil \), so \( G \) contains \( C_t \) for every \( t \in [3, \left\lceil \frac{n}{k-1} \right\rceil] \) and satisfies all three conditions.

Now suppose that \( G \) contains a non-bipartite block \( B \) with at least \( 2\delta - 13k \) vertices.

Let \( V \) be a minimal set of vertices such that every component of \( G[B - V] \) is 3-connected and has minimum degree \( \delta - 2k \). Observe that we could create such a set by sequentially identifying and removing pairs of vertices of \( B \) whose removal increases the number of components of \( G \). If \( 2k \) vertices have been removed, the number of components must be at least \( k + 1 \), and since only \( 2k \) vertices have been removed the minimum degree of what remains is at least \( \delta - 2k \). It follows that \( (k + 1)(\delta - 2k + 1) \leq n \), which is not true. Thus the procedure must have terminated with \( |V| < 2k \).

If \( G[B - V] \) has a component \( X \) which is not bipartite, then either it contains at least \( 2\delta - 15k \) vertices and by Theorem 2 we have \( ec(G), oc(G) \geq 2\delta - 15k - 1 \), or \( |X| < 2\delta - 15k \). In this second case, let \( Y \) be another component of \( G[B - V] \). If \( |Y| \geq 2\delta - 6k \) then let \( Q \) be a longest cycle in \( Y \). There exist vertex-disjoint paths \( P_1 \) and \( P_2 \) from \( X \) to \( Q \); let \( P \) be
the longer of the two possible paths formed from $P_1$, $P_2$ and part of $Q$. Then $P$ has both endpoints in $X$ and at least $\delta - 3k$ vertices. If $|Y| < 2\delta - 6k$, let $P_1$ and $P_2$ be vertex-disjoint paths from $X$ to $Y$; by Lemma 7 we can find a path on $|Y| - 2 \geq \delta - 2k - 2$ vertices linking $P_1$ and $P_2$, and again $|P| \geq \delta - 3k$. Finally by Lemma 7 applied to the endpoints of $P_1$ and $P_2$ in $X$, we can find both an even and an odd path of length at least $\delta - 5k$ in $X$ linking the endpoints of $P_1$ and $P_2$; so $oc(G), ec(G) \geq 2\delta - 10k$.

If every component of $G[B - V]$ is bipartite, let $Q$ be an odd cycle in $B$. Let $X$ be a component of $G[B - V]$. By either examining $Q - X$ (if $Q$ meets $X$ in two or more vertices) or by using 2-connectivity of $G[B]$ to construct two vertex disjoint paths from $Q$ to $X$ (if not) we can find a path $P$ with an even number of vertices whose endvertices $u$ and $v$ lie in the same bipartition class of $X$ and with at most one interior vertex $x$ contained in $X$. Let $H$ be the graph obtained by adding the edge $uv$ to $G[X - x]$. Then $\delta(H) \geq \delta - 2k - 1$, and $H$ is 2-connected and not bipartite. By Theorem 2, $oc(H) \geq 2\delta - 4k - 3$. Since every odd cycle in $H$ uses $uv$, on replacing $uv$ with $P$ in the longest odd cycle of $H$, we see that $oc(G) \geq 2\delta - 4k - 1$. Also by applying Theorem 2 to $X$ we see that $ec(G) \geq 2\delta - 4k$.

Applying Theorem 8 with $c = 1/k$ and $K = K(1/k)$ we have in both cases that $C_t$ is contained in $G$ for every even $4 \leq t \leq 2\delta - 16k - K$, and for every odd $K \leq t \leq 2\delta - 16k - K$. It remains only to show that in both cases $G$ also contains the requisite short odd cycles. Let $l$ be the smallest integer such that $\delta > \frac{2n}{2l+1}$. By Theorem 9, since $\delta(G) > \frac{2n}{2l+2j+1}$ for each $0 \leq j \leq K/2$ and since $oc(G) \geq 2\delta - 16k > K + l$, provided that $n_0 > 3K^4$, $G$ contains $C_{2l+2j-1}$ for each $0 \leq j \leq K/2$ as required.

Finally, suppose that $G$ contains a bipartite block $B$ with at least $2\delta - 13k$ vertices, and all its non-bipartite blocks have less than $2\delta - 13k$ vertices.

Let $X$ be $B$ with the cutvertices of $G$ sparse to $B$ removed, so that $\delta(X) \geq 9k$. Let $P$ be a path of maximum length in $X$. Observe that there are at most $k$ vertices of $X$ with less than $\delta - k$ neighbours in $X$; so if an endvertex $u$ of $P$ has $d_X(u) < \delta - k$ then there is a neighbour of $u$ whose successor $u'$ on $P$ has $d_X(u') \geq \delta - k$, and we obtain a path $P'$ with $V(P') = V(P)$ and endvertex $u'$. Similarly there are at most $k + 10k^2$ vertices of $X$ with less than $\delta$ neighbours in $X$ (the second term counting neighbours of cutvertices sparse to $B$), so by the same method we obtain a path $P''$ on the same vertex set as $P$ with an endvertex $v$ of degree at least $\delta$, all of whose neighbours are on $P''$ by maximality. By averaging, either there exists a neighbour $x$ of $v$ at distance at least $5\delta/2$ from $v$ along $P''$, or for every even $4 \leq t \leq 2\delta$ there exist neighbours $x$ and $y$ of $v$ such that the $x - y$ segment of $P''$ together with $v$ forms a copy of $C_t$. In the first case we have $ec(G) \geq 5\delta/2$, to which we can apply Theorem 8. Since $5\delta/2 - K > 2\delta$, in either case we have that $G$ contains $C_t$ for every even $4 \leq t \leq 2\delta$ and satisfies conditions (i) and (iii).

If $G$ contains a non-bipartite block $B'$, at least one of whose vertices is not a cutvertex, then as before by applying Lemma 6 to $B'$ with its sparse cutvertices removed we see that $G$ contains $C_t$ for every $3 \leq t \leq \delta + 1$ and so satisfies all three conditions. If $G$ has no non-bipartite blocks which are not composed entirely of cutvertices, then $G \in \mathcal{B}_k$ and again $G$ satisfies all three conditions. \qed
We note that although we have made no effort to optimise $n_0$, we can say $n_0 = O(k^{20})$, since in Theorem 8 we can take $K = O(k^5)$. It is certainly true that $n_0 = \Omega(k^2)$ is necessary, since there are graphs with $n$ vertices and minimum degree $\Theta(\sqrt{n})$ which do not contain $C_4$.

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References


