On the Burnside Semigroups $\boldsymbol{x}^n = \boldsymbol{x}^{n+m}$

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Abstract

In this paper we show that the congruence classes of A^* associated to the Burnside semigroup with $|A|$ generators defined by the equation $x^n = x^{n+m}$, for $n \geq 4$ and $m \geq 1$, are recognizable.
This problem was originally formulated by Brzozowski in 1969 for $m = 1$ and $n \geq 2$. Two years ago Aldo de Luca and Stefano Varricchio solved the problem for $n \geq 5$. A little later, John McCammond extended the problem for $m \geq 1$ and solved it independently in the cases $n \geq 6$ and $m \geq 1$. Our work, which is based on the techniques developed by de Luca and Varricchio, extends both these results.
We effectively construct a minimal generator Σ of our congruence. We introduce an elementary concept, namely the stability of productions, which allows us to eliminate all hypotheses related to the values of n and m. A substantial part of our proof consists of showing that all productions in Σ are stable for $n \geq 4$ and $m \geq 1$.
We also give an algorithm that solves the word problem and shows that the semigroup is finite $\mathcal{J}$-above. We prove that the frame of the $\mathcal{R}$-classes of the semigroup is a tree. We characterize also the $\mathcal{R}$-classes and the $\mathcal{D}$-classes of the semigroup and prove that its maximal subgroups are cyclic of order m in the cases $n \geq 4$ and $m \geq 1$.

1 Introduction

Let A be a finite alphabet with more than one letter, let A^* be the set of all words with letters in A (including the empty word 1) and let A^+ be the set $A^* \setminus \{1\}$.

In this work we will assume that n and m are fixed integers satisfying the restrictions $n \geq 2$, $m \geq 1$. The most important results are, however, obtained for $n \geq 4$ and $m \geq 1$.

Let π be the relation $\{(x^n, x^{n+m}), \forall x \in A^+\}$ on A^*. Take $\sim_\pi$ to be the smallest congruence that contains π, $\mathcal{M} \stackrel{\text{def}}{=} A^*/\sim_\pi$ and $\varphi : A^* \xrightarrow{\text{epi}} \mathcal{M}$ the canonical projection. Naturally the congruence $\varphi^{-1}\varphi$ is equal to $\sim_\pi$. The congruence class of a word $w \in A^*$ will be denoted by $[w]$. The Burnside semigroup with $|A|$ generators defined by the equation $x^n = x^{n+m}$ is the semigroup $(\mathcal{M} \setminus \{[1]\}, \cdot)$, where the operation $\cdot$ is the operation induced by the concatenation in A^*.

In these terms the conjecture of Brzozowski we deal with here can be formulated as follows:

$$\forall w \in A^*, [w] \text{ is recognizable.}$$

That is, for all words w there is a finite semigroup S and a morphism $f : A^* \longrightarrow S$ such that $f^{-1}f(w) = [w]$.

From an application of Thue-Morse words [11] and from a piece of work by Brzozowski et al. published in 1971 [2], we know that these Burnside semigroups are infinite. Even though we consider only the cases in which $n \geq 2$ and $m \geq 1$ in this work. It can be noted, however, that the idempotent semigroup — the Burnside semigroup for which $n = 1$ and $m = 1$ — is finite and completely known. From a classic result by Green and Rees [5], we also know that the semigroups in which $n = 1$ and $m \geq 1$ are finite if and only if the semigroups in which $n = 0$ and $m \geq 1$ — which are groups in this case — are finite. The problem of determining whether these groups, called Burnside groups, are finite is extremely complex and it is still being studied.

Brzozowski's conjecture was originally formulated for $m = 1$ in 1969 [1] and this is the case considered by de Luca and Varricchio. They proved the original conjecture for $n \geq 5$ in 1990 [3], thus remaining open the cases $n = 2$, $n = 3$ and $n = 4$. McCammond extended the conjecture for $m \geq 1$ in 1991 [8] and he independently presented a solution for the conjecture when $n \geq 6$ and $m \geq 1$.

Our work is deeply based on the techniques developed by de Luca and Varricchio. These authors suggest that refinements of their techniques might be enough to settle the conjecture in the case $n = 4$ and $m = 1$. However, they were more sceptical about the possibility that these techniques could help one solve the cases in which $n < 4$. In fact, we have not only proved the conjecture for $n \geq 4$ and $m \geq 1$ from a refinement of these techniques, but also opened a promising way for the demonstration of the cases in which $n = 3$ and $m \geq 1$ or even $n = 2$ and $m = 1$.

The main concept that allowed us such refinements is the concept of stability, which we now define. For any $\tau = (c, l) \in A^+ \times A^+$ we say that c is the *short* of τ while l is the *long* of τ. We respectively denote them by c_τ and l_τ. This terminology stems from the fact that in this work, in all explicit cases, we will have $|c_\tau| < |l_\tau|$. Let Ω be the subset of $A^+ \times A^+$ defined as $\Omega \stackrel{\text{def}}{=} \{(c, l) : c$ is both a proper suffix and a proper prefix of l and $c^{-1}l$ is an m-power$\}$. Each element of Ω is called *production*. Notice that $\pi \subset \Omega$. For each production τ we define the *base* of τ as $\mathrm{base}(\tau) \stackrel{\text{def}}{=} (|l_\tau| - |c_\tau|)/m$. Notice that $\mathrm{base}(\tau)$ is a period of c_τ and of l_τ. We say that the production τ is *stable* when $\mathrm{base}(\tau)$ is exactly the smallest period of c_τ and of l_τ.

Based on de Luca and Varricchio's techniques, we shall effectively construct a relation $\Sigma \subset \Omega$ (with a complex description and incomparable with π) such that the smallest congruence that contains it is $\sim_\pi$. This construction is made in Section 3, after some definitions and propositions. In our main results, we have as much as possible tried to prove them independently of one another. Also we tried to rely as little as possible on additional restrictions on n and m (remember that $n \geq 2$ and $m \geq 1$). This methodology originated the concept of stability of productions defined above. Although it is elementary, this is the most important key we found for a complete restructure and extension of de Luca and Varricchio's original proof. In fact, none of our results assumes any additional restriction on n or m, besides $n \geq 2$ and $m \geq 1$. In some cases, however, we will assume that productions of Σ are stable. Our approach contrasts with de Luca and Varricchio's since in their work, for example, all of the main results and almost all of the auxiliary ones depend on $n \geq 4$ and, in some cases, on $n \geq 5$.

Our first main result is that the smallest congruence that contains Σ coincides with the smallest one that contains π, and this holds for $n \geq 2$ and $m \geq 1$ (it does not depend even on the stability of productions of Σ). This result is proved in Section 4. Our second main result is that the productions of Σ are stable when $n \geq 4$ and $m \geq 1$. This is proved in Section 5. We will show that there are productions of Σ that are not stable when $n = 2$ and $m \geq 2$. The cases when $n = 2$ and $m = 1$ and when $n = 3$ and $m \geq 1$ remain open. Our results suppose the stability of the productions of Σ but, as we have already mentioned, they do not suppose additional restrictions on n or m. In particular a proof of the conjecture for $n = 3$ and $m \geq 1$ or $n = 2$ and $m = 1$ could be obtained if we could prove that the productions of Σ are stable in these cases, too. We have some evidence of the stability of the productions of Σ when $n = 3$ and $m \geq 1$.

Next we show that in every class $[w']$ there is a unique shortest word w from which we obtain any word in $[w']$ by successive substitutions of the shorts of productions of Σ by their respective longs. Also we show that Σ is a minimal relation such that the smallest congruence that contains it is $\sim_\pi$. This is proved in Section 6.

At last, we prove in Section 7 that the Burnside semigroup is finite $\mathcal{J}$-above and we prove the Brzozowski's conjecture. This was also proved by de Luca and Varricchio and by McCammond in the cases studied by them. We also show that the frame of $\mathcal{R}$-classes is a tree and that there is a unique shortest word in the inverse image by φ from a given $\mathcal{R}$-class. This was also shown by Imre Simon in 1970 [10] for the cases corresponding to the original conjecture. McCammond showed that the maximal subgroups of the Burnside semigroups studied by him are cyclic of order m and we show this for the semigroups studied by us, too. We have some new results related to the structure of these semigroups. We show that the internal structure of an $\mathcal{R}$-class (particularly its size and its internal "transitions") is determined exclusively by the longest suffix of the unique shortest word described above, which is a short of a production of Σ (if there is no such suffix, the $\mathcal{R}$-class is trivial). We also show that the regular $\mathcal{D}$-classes are exactly those which contain the congruence class of the short of some production in Σ. We know that two of these congruence classes are in the same $\mathcal{D}$-class if and only if their respective productions have the same base and their shorts have smallest periods which are conjugate words. We also show that the irregular $\mathcal{H}$-classes are trivial.

We make constant changes in the definitions from de Luca and Varricchio's work in order to allow refinements in certain proofs, as well as to make the text more compact or just as a matter of style.

The space constraints of this paper do not allow us to present proofs of most of our results. A complete version of this paper containing all the proofs in detail is the subject of my Masters's thesis [12] elaborated under the supervision of Imre Simon.

2 Concepts and combinatorial properties

We define a *word w on an alphabet A* to be a finite sequence of elements of A. We say that the length $|w|$ of the sequence is the *length* of w. We let A^* be the set of the words on A and A^k be the set of words of length k on A. Given any two words u and v we represent their concatenation by uv. For $k \in \mathbb{N} \setminus \{0\}$, we say that a word w is a *k-power* of u, or that u is a k-*subpower* of w, when $w = u^k$. We also say that w is a k-power or simply a power. We say that w is *primitive* if the shortest subpower of w is w.

We shall define some functions from A^* to the subsets of A^* as follows: $\mathrm{Fat}(w) \stackrel{\mathrm{def}}{=} \{u \in A^*$ such that $w \in A^*uA^*\}$ is the set of the *factors* of w; $\mathrm{Pref}(w) \stackrel{\mathrm{def}}{=} \{u \in A^*$ such that $w \in uA^*\}$ is the set of the *prefixes* of w; and $\mathrm{Suf}(w) \stackrel{\mathrm{def}}{=} \{u \in A^*$ such that $w \in A^*u\}$ is the set of *suffixes* of w. For $v \in \mathrm{Suf}(u)$ we say that uv^{-1} is the prefix of u of length $|u| - |v|$. When $v \in \mathrm{Pref}(u)$ we say that $v^{-1}u$ is the suffix of u of length $|u| - |v|$. For simplicity, we define $uv^{-k} \stackrel{\mathrm{def}}{=} u(v^k)^{-1}$. We define the *overlap* of two words u and v as: $u \rightleftharpoons v \stackrel{\mathrm{def}}{=} \max(\mathrm{Suf}(u) \cap \mathrm{Pref}(v))$. In other words, the overlap is the word of longest length that is simultaneously a suffix of u and a prefix of v. We say that two words u e u' are *conjugate* if $\exists v, v' \in A^*$ such that $u = vv'$ and $u' = v'v$.

We say that $u \in A^+$ (possibly longer than w) is a *period* of w when w is a factor of some power of u. In this case, we also say that $i = |u|$ is a *period* of w. When this integer i is minimal, then we say that it is *the smallest period* of w and we write: $i = \mathrm{per}(w)$. We also say that the corresponding u is a smallest period of w. Notice that if u is period of w then u is period of any factor of w. If $u \in \mathrm{Suf}(w) \setminus \{1\}$ then u is period of w if and only if $wu^{-1} \in \mathrm{Suf}(w)$.

Theorem 2.1 was proved by Fine and Wilf in 1965 [4] and it will be, together with theorem 2.2, a basic tool in the proof of the Stability Theorem as well as in the proof of other propositions.

Theorem 2.1 (Fine and Wilf Theorem) *Let u and v be two suffixes of w that are periods of w. Admit that $|w| \geq |u| + |v| - \gcd(|u|, |v|)$. Let x be the suffix of w of length $\gcd(|u|, |v|)$. Then x is a period of w, $u = x^{|u|/|x|}$ and $v = x^{|v|/|x|}$.*

Theorem 2.2 *The set of the words for which u is a period is the same as the set of the words for which v is a period if and only if the smallest subpowers of u and v are conjugate.*

One can show that the set of the conjugate words of u is $\mathrm{Fat}(uua^{-1}) \cap A^{|u|}$ where a is the last letter of u, and its cardinality is $|u|$ if u is primitive.

3 Relations equivalent to π

3.1 Productions, stability and the order $\leq_{\mathrm{r}}$

Take a relation $\mathcal{T} \subseteq A^+ \times A^+$ and let τ be an element of $\mathcal{T}$. We define $\sim_{\mathcal{T}}$ as the smallest congruence that contains $\mathcal{T}$ and say that $\mathcal{T}$ *generates* $\sim_{\mathcal{T}}$. We say that τ is *necessary* in $\mathcal{T}$ when the smallest congruence that contains $\mathcal{T} \setminus \{\tau\}$ is a proper subset of $\sim_{\mathcal{T}}$. We define $l_{\mathcal{T}}$ as the set $\{l_\tau,\ \tau \in \mathcal{T}\}$ and $c_{\mathcal{T}}$ as the set $\{c_\tau,\ \tau \in \mathcal{T}\}$.

Let $\tau,\ \sigma \in A^+ \times A^+$ and let u be a word; we define the *concatenation* of τ and u as $\tau u \stackrel{\mathrm{def}}{=} (c_\tau u, l_\tau u)$ and the *concatenation* of u and τ as $u\tau \stackrel{\mathrm{def}}{=} (uc_\tau, ul_\tau)$. If u is a suffix of c_τ and of l_τ, we define $\tau u^{-1} \stackrel{\mathrm{def}}{=} (c_\tau u^{-1}, l_\tau u^{-1})$ and if u is a prefix of c_τ and of l_τ, we define $u^{-1}\tau \stackrel{\mathrm{def}}{=} (u^{-1}c_\tau, u^{-1}l_\tau)$. We define a partial order $\leq_{\mathrm{r}}$ on $A^+ \times A^+$ and its strict order $<_{\mathrm{r}}$ as follows: $\sigma \leq_{\mathrm{r}} \tau \stackrel{\mathrm{def}}{\iff} \exists u, v \in A^*$ such that $\tau = u\sigma v$; and

$\sigma <_{\mathrm{r}} \tau \overset{\mathrm{def}}{\iff} \sigma \leq_{\mathrm{r}} \tau,\ \sigma \neq \tau$. We define $\mathrm{irred}(\mathcal{T})$ as the subrelation of $\mathcal{T}$ consisting of the minimal of $\mathcal{T}$ under the order $\leq_{\mathrm{r}}$; the elements of $\mathrm{irred}(\mathcal{T})$ are called *irreducible* in $\mathcal{T}$.

Fixed a set $\mathcal{T}$ of productions and given ϵ in $A^+ \times A^+$ we say that ϵ is a *expansion* under $\mathcal{T}$ when $\exists \sigma \in \mathcal{T}$ such that $\epsilon \in A^* \sigma A^*$. We also say that the pair (l_ϵ, c_ϵ) is a *contraction* under $\mathcal{T}$. For a production τ we let $\mathrm{b_L}(\tau) \overset{\mathrm{def}}{=}$ the prefix of l_τ of length $\mathrm{base}(\tau)$ be the *left base* of τ; $\mathrm{b_R}(\tau) \overset{\mathrm{def}}{=}$ the suffix of l_τ of length $\mathrm{base}(\tau)$ be the *right base* of τ. Note that $\mathrm{b_R}(\tau)^m$, $\mathrm{b_L}(\tau)^m$, $\mathrm{b_R}(\tau)$ e $\mathrm{b_L}(\tau)$ are periods of l_τ and c_τ. Note also that $\pi \subset \Omega$ and, if $\tau = (x^n, x^{n+m})$ is a production of π, then $\mathrm{b_L}(\tau) = \mathrm{b_R}(\tau) = x$ and $\mathrm{base}(\tau) = |x|$. We say that two productions τ and σ are *conjugate*, when $\mathrm{b_R}(\tau)$ and $\mathrm{b_R}(\sigma)$ are conjugate words. The relation defined for the pairs of conjugate productions is an equivalence relation.

The following propositions give us an interesting set of properties of the partial order $\leq_{\mathrm{r}}$.

Proposition 3.1 *Let $\tau \in \Omega$ and let $\sigma \in A^+ \times A^+$ with $\sigma \leq_{\mathrm{r}} \tau$. Then $\sigma \in \Omega$ and τ and σ are conjugate.*

Proposition 3.2 *Let $\tau,\ \sigma \in \Omega$. Then $\sigma \leq_{\mathrm{r}} \tau \iff \mathrm{base}(\tau) = \mathrm{base}(\sigma)$ and $l_\sigma \in \mathrm{Fat}(l_\tau)$. If $|c_\sigma| \geq \mathrm{base}(\sigma)$, then $\sigma \leq_{\mathrm{r}} \tau \iff \mathrm{base}(\tau) = \mathrm{base}(\sigma)$ and $c_\sigma \in \mathrm{Fat}(c_\tau)$.*

Recall that a production τ is *stable* when $\mathrm{per}(c_\tau) = \mathrm{per}(l_\tau) = \mathrm{base}(\tau)$. Some properties related to this concept follow.

Proposition 3.3 *Every production $\tau \in \Omega$ satisfies $\mathrm{per}(c_\tau) \leq \mathrm{per}(l_\tau) \leq \mathrm{base}(\tau)$. So τ is stable if and only if $\mathrm{per}(c_\tau) = \mathrm{base}(\tau)$.*

Proposition 3.4 *Let τ be a stable production. Then $\mathrm{b_R}(\tau)$ is primitive.*

From proposition 3.4, it is natural to ask if the converse holds. The answer is no. In proposition 3.5 we will see a sufficient condition for this converse to be true.

Proposition 3.5 *Let $\tau \in \Omega$ be such that $\mathrm{b_R}(\tau)$ is primitive. If $|c_\tau| \geq 2\mathrm{base}(\tau)$ then τ is stable.*

3.2 Why π is not a "good" generator of $\sim_\pi$

We will present now some "shortcomings" of the relation π. We will show that there are productions in π that are not minimal in $\sim_\pi$ according the order $\leq_{\mathrm{r}}$ and there are some that are direct consequences of other productions of π (they are not necessary in π). Note the following examples.

Example 3.6 *Let $\tau = (x^n, x^{n+m}) \in \pi$ in which $x = (b(ab)^{n+m-1})$. Let σ be the production $((ab)^n x^{n-2} (ba)^n, (ab)^n x^{n+m-2} (ba)^n)$. Then we have that $\sigma <_{\mathrm{r}} \tau$ and $\sigma \in \sim_\pi$, although σ is not of the form (y^n, y^{n+m}) for any $y \in A^+$. We have also that σ is not stable when $n = 2$ and $m \geq 2$.*

Proof. Note that $((ab)^n, (ab)^{n+m}) = ((ab)^n, ax) \in \pi$ and note that $((ba)^n, (ba)^{n+m}) = ((ba)^n, xa) \in \pi$. Note that $(ab)^n x^{n-2} (ba)^n \sim_\pi (ab)^{n+m} x^{n-2} (ba)^{n+m} = axx^{n-2}xa \sim_\pi axx^{n+m-2}xa = (ab)^{n+m} x^{n+m-2} (ba)^{n+m} \sim_\pi (ab)^n x^{n+m-2} (ba)^n$, from which we have that $\sigma \in \sim_\pi$. Then we have that $\tau = b(ab)^{m-1}\,\sigma\, b(ab)^{m-1}$, that $\sigma <_{\mathrm{r}} \tau$, that $\sigma \in \sim_\pi$, and that σ is not of the form (y^n, y^{n+m}) for any $y \in A^+$.

Suppose that $n = 2$ and that $m \geq 2$. Then $c_\sigma = ababbaba$ and $\mathrm{per}(c_\sigma) = 5 < 3+2m = |x| = \mathrm{base}(\tau) = \mathrm{base}(\sigma)$ due to proposition 3.1. Then we have that σ is not stable in these cases. □

Example 3.7 *Let $x = aa$ and let $\tau = (x^n, x^{n+m}) \in \pi$. Then the production τ is neither stable nor is necessary in π.*

Example 3.8 *Let $\tau = (x^n, x^{n+m}) \in \pi$ be the production in which $x = c(ab)^{n+m}d$. Then the production τ is stable but it is not necessary in π.*

Our first aim is to construct a relation $\Sigma \subseteq \Omega$ that has the following properties:

- The smallest congruence that contains π is the same as the one that contains Σ.
- The productions of Σ are all irreducible in $\sim_\Sigma$.
- All the productions of Σ are necessary in Σ.

These properties will be proved later (in the Equivalence Theorem and in the Expansibility Theorem). They give us an interesting intuition of what this relation Σ is. We also show that all the words from a given class $[w]$ can be obtained by expansions under Σ from the smallest word in $[w]$. This fact does not apply to expansions under π. Example 3.6 shows this, for expansions and contractions under π are used to show that $c_\sigma \sim_\pi l_\sigma$. De Luca and Varricchio's work uses a superset of this Σ. They do not work with the concept of necessity.

3.3 Reductions and the construction of Σ

We will give some definitions in order to construct an operation on the set of relations with productions in Ω. This operation will be used in the construction of Σ. Given $\mathcal{T} \subseteq \Omega$ and $\tau,\ \sigma \in \mathcal{T}$ we define the relation $\mathrm{R_R}$ as

$$\tau \ \mathrm{R_R}\ \sigma \ \overset{\mathrm{def}}{\iff} \ |c_\tau \rightleftharpoons l_\sigma| > |c_\sigma| \ \text{ and } \ \mathrm{base}(\tau) > \mathrm{base}(\sigma).$$

Note that $c_\sigma, c_\tau \rightleftharpoons l_\sigma \in \mathrm{Pref}(l_\sigma)$ and that if $\tau\ \mathrm{R_R}\ \sigma$ then we have that $c_\sigma \in \mathrm{Pref}(c_\tau \rightleftharpoons l_\sigma)$. We define a function that applies a pair of productions in A^*: $\mathrm{m_R}(\tau,\sigma) \overset{\mathrm{def}}{=} {c_\sigma}^{-1}(c_\tau \rightleftharpoons l_\sigma)$ when $\tau\ \mathrm{R_R}\ \sigma$ and 1 otherwise. Note that $\mathrm{m_R}(\tau,\sigma) \in \mathrm{Suf}(c_\tau \rightleftharpoons l_\sigma) \subseteq \mathrm{Suf}(c_\tau) \subseteq \mathrm{Suf}(l_\tau)$. Then we can define the binary operation $\otimes_{\mathrm{dir}}$ in Ω as

$$\tau \otimes_{\mathrm{dir}} \sigma \ \overset{\mathrm{def}}{=} \ \tau\,(\mathrm{m_R}(\tau,\sigma))^{-1} = (c_\tau\,(\mathrm{m_R}(\tau,\sigma))^{-1}, l_\tau\,(\mathrm{m_R}(\tau,\sigma))^{-1}),$$

and we say that $\tau \otimes_{\mathrm{dir}} \sigma$ is the *right reduction* of τ by σ. In a dual way, we can define the relation $\mathrm{R_L}$ as

$$\tau \ \mathrm{R_L}\ \sigma \ \overset{\mathrm{def}}{\iff} \ |l_\sigma \rightleftharpoons c_\tau| > |c_\sigma| \ \text{ and } \ \mathrm{base}(\tau) > \mathrm{base}(\sigma),$$

$\mathrm{m_L}(\tau,\sigma) \overset{\mathrm{def}}{=} (l_\sigma \rightleftharpoons c_\tau){c_\sigma}^{-1}$ when $\tau\ \mathrm{R_L}\ \sigma$ and 1 otherwise, and, finally, we define the *left reduction* of τ by σ as

$$\sigma \otimes_{\mathrm{esq}} \tau \ \overset{\mathrm{def}}{=} \ (\mathrm{m_L}(\tau,\sigma))^{-1}\,\tau = ((\mathrm{m_L}(\tau,\sigma))^{-1}c_\tau, (\mathrm{m_L}(\tau,\sigma))^{-1}l_\tau).$$

Now we define the operation in question. For $\mathcal{T} \subseteq \Omega$ let $\otimes(\mathcal{T})$ denote the closure of $\mathcal{T}$ under the operations $\otimes_{\mathrm{dir}}$ and $\otimes_{\mathrm{esq}}$. We say that $\otimes(\mathcal{T})$ is the *closure under reductions* of $\mathcal{T}$. We can show that $\otimes(\mathcal{T}) \subseteq \sim_{\mathcal{T}}$.

Example 3.6 is illustrative. For $\sigma' = ((ab)^n, (ab)^{n+m})$ and for $\sigma'' = ((ba)^n, (ba)^{n+m})$, we can see that $\tau \ \mathrm{R_R}\ \sigma''$, that $\tau \ \mathrm{R_L}\ \sigma'$, and that $\sigma = (\sigma' \otimes_{\mathrm{esq}} \tau) \otimes_{\mathrm{dir}} \sigma'' = \sigma' \otimes_{\mathrm{esq}} (\tau \otimes_{\mathrm{dir}} \sigma'')$.

Some propositions related to these concepts follow below.

Proposition 3.9 *Let $\tau,\ \sigma \in \Omega$. Then $\tau \otimes_{\mathrm{dir}} \sigma \in \Omega$, $\mathrm{base}(\tau \otimes_{\mathrm{dir}} \sigma) = \mathrm{base}(\tau)$ and $\tau \otimes_{\mathrm{dir}} \sigma \leq_{\mathrm{r}} \tau$, the inequality being strict iff $\tau \ \mathrm{R_R}\ \sigma$.*

We will construct other relations which are equivalent to π. Note that we implicitly obtain an algorithm that exhibits the productions of Σ that have base less than or equal to a fixed integer, as a consequence of the definitions. In order to understand these definitions better, remember that if $\tau \otimes_{\mathrm{dir}} \sigma \neq \tau$, then $\tau \otimes_{\mathrm{dir}} \sigma <_{\mathrm{r}} \tau$ and $\mathrm{base}(\sigma) < \mathrm{base}(\tau) = \mathrm{base}(\tau \otimes_{\mathrm{dir}} \sigma)$ because of propositions 3.9.

Let $\Sigma_0 = \emptyset$. Then we define:

$$\begin{aligned}
\pi_i &\stackrel{\mathrm{def}}{=} \{(x^n, x^{n+m}), \forall x \in A^i\} \\
\pi'_i &\stackrel{\mathrm{def}}{=} \{\tau \in \pi_i \text{ such that } \mathrm{Fat}(l_\tau) \cap l_{\Sigma_{\lfloor \frac{i-1}{m} \rfloor}} = \emptyset\} \\
\Sigma'_i &\stackrel{\mathrm{def}}{=} \otimes(\pi'_i \cup \Sigma_{i-1}) \\
\Sigma_i &\stackrel{\mathrm{def}}{=} \mathrm{irred}(\Sigma'_i).
\end{aligned}$$

We also define the sets:

$$\begin{aligned}
\pi &= \cup_{i=1}^{\infty} \pi_i = \{(x^n, x^{n+m}), \forall x \in A^+\} \\
\pi' &\stackrel{\mathrm{def}}{=} \cup_{i=1}^{\infty} \pi'_i \\
\Sigma' &\stackrel{\mathrm{def}}{=} \cup_{i=1}^{\infty} \Sigma'_i \\
\Sigma &\stackrel{\mathrm{def}}{=} \cup_{i=1}^{\infty} \Sigma_i = \mathrm{irred}(\Sigma') \\
\Sigma'' &\stackrel{\mathrm{def}}{=} A^* \Sigma A^* \cap \Omega.
\end{aligned}$$

It is interesting to note that $\pi_1 = \pi'_1 = \Sigma'_1 = \Sigma_1 = \{(a^n, a^{n+m}), \text{ for } a \in A\}$. We can see that π_i and π'_i are the subsets obtained by all the productions of π and π', respectively, that have base equal to i. On the other hand, Σ_i is the subset of Σ consisting of all productions of Σ that have base less than or equal to i. We can also show that $\Sigma'' = \cup_{\tau \in \Sigma}(\mathrm{Suf}(\mathrm{b_L}(\tau)^*)\, \tau\, \mathrm{Pref}(\mathrm{b_R}(\tau)^*))$ and that $\Sigma \subset \Sigma' \subset \Sigma''$. In other words, a production in Σ'' is obtained by a "coherent" concatenation of a production of Σ with more periods associated to this production. We will understand the importance of this new relation in propositions 6.4 and 6.5, when we will have defined the concept of expander.

Proposition 3.10 *If the productions of Σ are stable then the productions of Σ'' are stable.*

In the following proposition we will see that the productions of Σ not only are irreducible in Σ' but also cannot suffer right nor left strict reductions by productions of Σ.

Proposition 3.11 *Let $\tau,\ \sigma \in \Sigma$. Then $(\tau, \sigma) \notin \mathrm{R_R} \cup \mathrm{R_L}$.*

4 Equivalence between Σ and π

The theorem below shows the equivalence between the relations Σ and π. Note that no additional restriction (besides $n \geq 2$ e $m \geq 1$) is made.

Theorem 4.1 (Equivalence Theorem) *The relations π and Σ generate the same congruence.*

Proof. We will just give an idea of the proof. We can show quite easily that Σ, Σ' and π' generate the same congruence. In this way, the difficulty lies in the proof that π and π' generate the same congruence. For this case, it is sufficient to show that $\pi \setminus \pi' \subseteq \sim_{\pi'}$. If we analyse the construction of π', we see that if $\tau \in \pi \setminus \pi'$ then there is a production σ in Σ with smaller base than the base of τ such that l_σ is a factor of l_τ. Then we have two alternatives. First, τ is not stable and it is easy to see that $\mathrm{b_R}(\tau)$ is not primitive. Using a similar idea to the proof of example 3.7, we see that τ is not a necessary production in π. In the second alternative, τ is stable. In this last case, the general idea of the proof is the following: let u be a word such that $\tau = (u^n, u^{n+m})$. By an induction on $\mathrm{base}(\tau)$, we can prove that $\sigma \in \sim_{\pi'}$ and consequently $l_\sigma \sim_{\pi'} c_\sigma$. Using the Fine and Wilf Theorem, we can show that $l_\sigma \in \mathrm{Fat}(u^3)$. The simplest case is that in which l_σ is a factor of u (see example 3.8). In this case we can contract u to a word u', with smaller length than u, substituting the occurrence of l_σ by c_σ. So we will have, by induction on $\mathrm{base}(\tau)$, that $({u'}^n, {u'}^{n+m}) \in \sim_{\pi'}$. From this we obtain that $u^{n+m} \sim_{\pi'} {u'}^{n+m} \sim_{\pi'} {u'}^n \sim_{\pi'} u^n$ and, consequently, we obtain that $\tau \in \sim_{\pi'}$. □

5 The Stability Theorem

The main result in this section is the Stability Theorem. We prove that every production of Σ (and also of Σ' and of Σ'') is stable when $n \geq 4$ and $m \geq 1$.

Proposition 5.1 gives us a set of very interesting properties of Σ'' and productions of Σ. It will be used in the proof of the Stability Theorem later on and also of the Expansibility Theorem and other propositions.

Proposition 5.1 *Let τ, $\sigma \in \Sigma''$ be two stable productions. If $l_\sigma \in \mathrm{Fat}(l_\tau)$, then $\sigma \leq_{\mathrm{r}} \tau$.*

Note that this implies that there are no two distinct productions τ, $\sigma \in \Sigma$ such that l_σ is a factor of l_τ. We are now going to see some lemmas that will help us in the proof of the Stability Theorem.

Lemma 5.2 *If ρ, σ, σ' are productions such that $\sigma' \otimes_{\mathrm{esq}} \rho$ and $\rho \otimes_{\mathrm{dir}} \sigma$ are stable productions, then $(\sigma' \otimes_{\mathrm{esq}} \rho) \otimes_{\mathrm{dir}} \sigma = \sigma' \otimes_{\mathrm{esq}} (\rho \otimes_{\mathrm{dir}} \sigma)$.*

For the next three lemmas, we suppose the following. We have productions $\rho \in \pi'$, $\tau \in \Sigma'$ and a non-empty sequence $\{\sigma_i\}$ of length k, which is minimal, of stable productions in Σ whose successive right reductions on ρ results in τ.

Lemma 5.3 *In the situation described above, we have that $\mathrm{base}(\sigma_k) > \beta\mathrm{base}(\sigma_{k-1}) > \cdots > \beta^{k-1}\mathrm{base}(\sigma_1)$, where $\beta = \max(\{1, \varphi - 1\})$ and φ is such that $|c_\sigma| \geq \varphi\mathrm{base}(\sigma)$ for every σ in the sequence $\{\sigma_i\}$. Moreover, $c_{\sigma_i} \in \mathrm{Fat}(l_{\sigma_{i+1}})$, for $i = 1, 2, \ldots, k-1$.*

Lemma 5.4 *Let* $\sigma \in \Sigma$ *be a stable production such that* $\mathrm{base}(\sigma) > \mathrm{base}(\sigma_k)$ *and let* $x \in A^*$ *be such that* $xc_\sigma \in \mathrm{Pref}(c_\rho)$. *Then* $|xc_\sigma| \leq |c_\tau|$.

Proof. We shall give some ideas of the proof. Given a sequence of right reductions there may not be a stable production σ with base longer than that of σ_k such that its short is a factor of c_ρ and that it ends after c_τ, for this would signify that this σ was also reduced by some production in $\{\sigma_i\}$. By proposition 3.9 we would have that these two productions would relate to each other according to relation $\mathrm{R_R}$. This contradicts proposition 3.11. □

Lemma 5.5 *Let* $\eta \in A^*$ *be such that* $\rho = \tau\eta$. *Then* $|\eta| < \mathrm{base}(\tau)$.

Proof. We have that c_{σ_k} is a suffix of c_τ. Suppose, for a contradiction, that $|\eta| \geq |\mathrm{b_R}(\rho)|$. Then we have that there is an occurrence of c_{σ_k} in c_ρ which ends $\mathrm{base}(\tau)$ letters on the right of the end of the first occurrence of c_τ in c_ρ. Suppose $k > 1$. Then $\mathrm{base}(\sigma_k) > \mathrm{base}(\sigma_{k-1})$ by lemma 5.3. Let τ' be the production obtained by right reductions from ρ by the productions of the sequence $\{\sigma_i\}$ except the production σ_k. So $\tau = \tau' \otimes_{\mathrm{dir}} \sigma_k$. Applying lemma 5.4 to τ' we have that this occurrence could not end after the first occurrence of $c_{\tau'}$ in c_ρ. For $k = 1$ this is also verified because we take ρ as τ'. Whatever the case is, we have that this occurrence of c_{σ_k} must be inside $c_\rho \rightleftharpoons l_{\sigma_k}$ (note that this overlap is a suffix of $c_{\tau'}$). This implies that $|c_\rho \rightleftharpoons l_{\sigma_k}| \geq \mathrm{base}(\tau) + |c_{\sigma_k}|$. As $|c_{\sigma_k}| \geq \mathrm{base}(\sigma_k)$, for σ_k is stable, using the Fine and Wilf Theorem, we conclude that $\mathrm{b_R}(\rho)$ is not primitive: but it can not happen since $\rho \in \pi'$. □

Next theorem shows us that the productions of Σ are stable when $n \geq 4$ and $m \geq 1$. However, it is interesting to notice that for $n = 2$ and for $m \geq 2$, not all productions in Σ are stable. To show this, note that σ in example 3.6 is an unstable production for such values of n and m. This σ is an element of Σ' and there exists an element of Σ that is less than or equal to σ under the order $\leq_{\mathrm{r}}$, which is not stable either.

Theorem 5.6 (Stability Theorem) *Admit* $n \geq 4$ *and* $m \geq 1$. *Then every production of* Σ *is stable.*

Proof. We shall prove that the productions of Σ' are stable. Let $\tau \in \Sigma'$. Our proof is by double induction. The external induction is on $\mathrm{base}(\tau)$ and the internal one is on $n\mathrm{base}(\tau) - |c_\tau|$.

When $\mathrm{base}(\tau) = 1$, we have that $\tau = (a^n, a^{n+m})$ for some $a \in A$ and $\tau \in \pi'$. When $\tau \in \pi'$, we have that $n\mathrm{base}(\tau) - |c_\tau| = 0$. As all productions of π' are stable (using the Fine and Wilf Theorem), we already have our induction basis. We will suppose, from now on, that $\tau \notin \pi'$. Note that, by the construction of Σ, if $\tau \in \Sigma'$ then there is a $\rho \in \pi'$ from which successive reductions were made, on the right and/or on the left, until we obtain τ. We can suppose $\{\sigma_i\}$ is a shortest sequence of productions $\{\sigma_i\}$ with which the reductions from τ were made. The productions in $\{\sigma_i\}$ are stable by induction hypothesis — their bases are less than $\mathrm{base}(\tau)$ by proposition 3.9. Also, there are words μ, η such that $\rho = \mu\tau\eta$. Our aim is to show that the productions of Σ are stable, but for this we show that $|\mu|, |\eta| < \mathrm{base}(\tau)$. In this case, we will have that $|c_\tau| = |\mu^{-1} c_{\tau_0} \eta^{-1}| \geq 2\mathrm{base}(\tau)$ and, by using proposition 3.5, we will have that τ is stable, proving the Stability Theorem.

From proposition 5.2, using the internal induction hypothesis, we see that a right reduction commutes with a left one. This implies that we can do all the right reductions first and then all the left ones. To show that $|\eta| < \mathrm{base}(\tau)$ it is sufficient to analyse the case in which only right reductions are made. Using lemma 5.5, we have that $|\eta| < \mathrm{base}(\tau)$.

In a dual way, we also have that $|\mu| < \mathrm{base}(\tau)$. □

6 On the expansibility of Σ

Proposition 6.1 *Let $\tau,\ \sigma \in \Sigma$ be two stable productions. Then we have that $c_\tau \rightleftharpoons c_\sigma = l_\tau \rightleftharpoons c_\sigma = c_\tau \rightleftharpoons l_\sigma = l_\tau \rightleftharpoons l_\sigma$ if and only if τ and σ are non-conjugate productions.*

Propositions 6.1 and 5.1 are extremely important in the proof of lemma 6.2 and, consequently, in the proof of the Expansibility Theorem.

Lemma 6.2 *Let $\mu, \eta, \mu', \eta' \in A^*$, let $\tau, \sigma \in \Sigma$ be two stable productions and let $\epsilon = (w', w'') = \mu'\sigma\eta'$ and $\epsilon' = (w''', w'') = \mu\tau\eta$ be two different expansions such that $l_\epsilon = l_{\epsilon'}$. Then there is $w'''' \in A^+$ such that $\tau \leq_{\mathrm{r}} \epsilon'' = (w'''', w')$ and $\sigma \leq_{\mathrm{r}} \epsilon''' = (w'''', w''')$ where ϵ''' and ϵ'' are two different expansions such that $c_{\epsilon''} = c_{\epsilon'''}$.*

The above lemma gives a property very similar to that of local confluence, studied in the Theory of Rewriting Systems. In fact, one can show that the contractions under Σ induce a rewriting system with this property. This gives a proof of the item 5 and of the item 3 of the Expansibility Theorem. One can find the definitions of the main concepts involved in this theory in [6] and an interesting application in the proof of the finiteness of the idempotent semigroup in [9].

Theorem 6.3 (Expansibility Theorem) *Suppose that the productions of Σ are stable. Let w' be any word. Then the following assertions hold.*

1. *The productions of Σ are necessary in Σ.*
2. *The productions of Σ are irreducible in $\sim_\pi$.*
3. *There is a unique word with minimal length in $[w']$.*
4. *A word w is a word with minimal length in $[w']$ if and only if no factor of w is the long of some production of Σ.*
5. *There is a sequence s of expansions under Σ that expands w to w', where w is the word with minimal length in $[w']$.*

As a consequence of proposition 6.4, we can define the *expander* of a given expansion ϵ by $\exp(\epsilon) \stackrel{\mathrm{def}}{=} \max_{\leq_{\mathrm{r}}}(\{\rho \in \Sigma'' \text{ such that } \rho \leq_{\mathrm{r}} \epsilon\})$. The usefulness of the definition of Σ'' lies on the fact that the expander of an expansion is unique, while there may be several productions of Σ (all of them conjugate to each other, as a matter of fact), that originate the same expansion.

Proposition 6.4 *Let ϵ be an expansion under Σ, let $\tau,\ \sigma \in \Sigma''$ and let $\mu, \eta, \mu', \eta' \in A^*$ be such that $\epsilon = \mu\tau\eta = \mu'\sigma\eta'$. Then τ and σ are conjugate productions and $\rho = \min(\{\mu, \mu'\})^{-1}\epsilon\min(\{\eta, \eta'\})^{-1}$ is such that $\tau,\ \sigma \leq_{\mathrm{r}} \rho$ and $\rho \in \Sigma''$.*

Given an expansion $\epsilon = (w, w')$, the proposition 6.5 gives us another interpretation for the expander: if $\sigma = \exp(\epsilon)$, then c_σ and l_σ are, respectively, "maximal" factors of w and w' that have period $\mathrm{base}(\epsilon)$; it also allows us to identify the prefix and the suffix common to w and to w' of maximal length.

Proposition 6.5 *Let $\sigma \in \Sigma''$. Let $\epsilon = u\sigma v$ be an expansion under Σ. Consider the following assertions.*

1. $\sigma = \exp(\epsilon)$,
2. $\mathrm{Pref}(c_\epsilon) \cap \mathrm{Pref}(l_\epsilon) = \mathrm{Pref}(uc_\sigma)$ *and* $\mathrm{Suf}(c_\epsilon) \cap \mathrm{Suf}(l_\epsilon) = \mathrm{Suf}(c_\sigma v)$,
3. *Let $u' \in \mathrm{Suf}(u)$ and $v' \in \mathrm{Pref}(v)$. Then $\mathrm{b_R}(\sigma)$ is a period of $u' l_\sigma v'$ implies that $u' = v' = 1$.*
4. *Let $u' \in \mathrm{Suf}(u)$ and $v' \in \mathrm{Pref}(v)$. Then $\mathrm{b_R}(\sigma)$ is a period of $u' c_\sigma v'$ implies that $u' = v' = 1$.*

Then we have that $(1) \iff (2) \iff (3) \Longleftarrow (4)$. *Moreover* (3) *implies* (4) *when* $|c_\sigma| \geq |\mathrm{base}(\sigma)|$.

7 The Structure of the Burnside Semigroup

7.1 Green relations

Let $(S, \cdot)$ be a semigroup. We define S^1 to be S if S is a monoid and S together a neutral element otherwise. Given also $u, v \in S$, we define the quasi-orders and Green relations as follows.

$$
\begin{array}{rcl@{\qquad}rcl}
v \leq_\mathcal{R} u & \stackrel{\text{def}}{\iff} & \exists x \in S^1 \text{ such that } v = u \cdot x & v <_\mathcal{R} u & \stackrel{\text{def}}{\iff} & v \leq_\mathcal{R} u \not\mathcal{R}\, v \\
v \leq_\mathcal{L} u & \stackrel{\text{def}}{\iff} & \exists x \in S^1 \text{ such that } v = x \cdot u & v <_\mathcal{L} u & \stackrel{\text{def}}{\iff} & v \leq_\mathcal{L} u \not\mathcal{L}\, v \\
v \leq_\mathcal{J} u & \stackrel{\text{def}}{\iff} & \exists x, y \in S^1 \text{ such that } v = x \cdot u \cdot y & v <_\mathcal{J} u & \stackrel{\text{def}}{\iff} & v \leq_\mathcal{J} u \not\mathcal{J}\, v \\
v \,\mathcal{R}\, u & \stackrel{\text{def}}{\iff} & v \leq_\mathcal{R} u \leq_\mathcal{R} v & v \,\mathcal{J}\, u & \stackrel{\text{def}}{\iff} & v \leq_\mathcal{J} u \leq_\mathcal{J} v \\
v \,\mathcal{L}\, u & \stackrel{\text{def}}{\iff} & v \leq_\mathcal{L} u \leq_\mathcal{L} v & v \,\mathcal{H}\, u & \stackrel{\text{def}}{\iff} & v \,\mathcal{R}\, u \,\mathcal{L}\, v \\
v \,\mathcal{D}\, u & \stackrel{\text{def}}{\iff} & \exists x \in S^1 \text{ such that } v \,\mathcal{L}\, x \,\mathcal{R}\, u & & &
\end{array}
$$

The relations $\mathcal{R}, \mathcal{L}, \mathcal{J}, \mathcal{H}, \mathcal{D}$ are all equivalence relations, and $\mathcal{R}$ is a left congruence while $\mathcal{L}$ is a right congruence. It can be easily shown that $\mathcal{R}, \mathcal{L} \subseteq \mathcal{D} \subseteq \mathcal{J}$. For $\rho \in \{\mathcal{R}, \mathcal{L}, \mathcal{J}\}$, we also define the *set of elements ρ-above* of u to be $\mathrm{Up}_\rho(u) \stackrel{\text{def}}{=} \{w \in S \text{ such that } u \leq_\rho w\}$. We say that the semigroup s is finite ρ-above when $\mathrm{Up}_\rho(u)$ is finite for every $u \in S$. Given $w \in S$, we say that the $\mathcal{R}$-class (or $\mathcal{L}$-class) of w is *rigid* if $w \cdot u <_\mathcal{R} w$ ($u \cdot w <_\mathcal{L} w$) for all $u \in S^1 \setminus \{1\}$. We say that the $\mathcal{D}$-class of w is *rigid* if its $\mathcal{R}$-classes and $\mathcal{L}$-classes are rigid. We say that a subset of S is *trivial* if it has only one element. Note that all rigid $\mathcal{R}$-classes, all rigid $\mathcal{L}$-classes and all rigid $\mathcal{D}$-classes are trivial. Note also that the converse does not hold.

We say that an element e of a semigroup is *idempotent* when $e \cdot e = e$ and we say that an element u is *regular* when there is $v \in S$ such that $u \cdot v \cdot u = u$ and u is *irregular* otherwise. The fact that an element is regular recovers partially the existence of an inverse element, as in groups. It is known that in the same $\mathcal{D}$-class there may not be irregular and regular elements. Thus the $\mathcal{D}$-classes may be classified in regular or irregular according to the regularity of their elements. The reader is referred to [7] for more details.

7.2 A monoid isomorphic to Burnside's

If we suppose that the productions of Σ are stable, according to item 3 of the Expansibility Theorem, we can define the *representative* $\text{rep}(w)$ of a given word w to be the shortest word of $[w]$. Given a set of words W, we let $\text{rep}(W)$ be the set $\{\text{rep}(w),\ w \in W\}$ and we can define $\mathcal{M}'$ as the set $\text{rep}(A^*)$, or, in other words, $\mathcal{M}' = \{w \in A^*$ such that $\nexists \sigma \in \Sigma$ with $l_\sigma \in \text{Fat}(w)\}$ by item 4 of the Expansibility Theorem. We have that the representative function is a bijection between the elements of $\mathcal{M}$, the Burnside monoid defined in the section 1, and of $\mathcal{M}'$. Note that every factor of an element of $\mathcal{M}'$ is an element of $\mathcal{M}'$. Thus the operation $\cdot$ on $\mathcal{M}$ induces an operation on $\mathcal{M}'$, and we have that $u \cdot v = \text{rep}([u] \cdot [v]) = \text{rep}([uv]) = \text{rep}(uv)$. It is easy to verify that $u_1 \cdot u_2 \cdot \dots \cdot u_k = \text{rep}(u_1 u_2 \cdots u_k)$. Thus we have that the monoid $\mathcal{M}'$ is isomorphic to the Burnside monoid $\mathcal{M}$. By using the fact that no production of π has the word 1 as its short or as its long, it is easy to verify that $[1] = \{1\}$ and also that the $\mathcal{D}$-class of 1, relative to the monoid $\mathcal{M}'$, is rigid. By defining $\mathcal{S}' \stackrel{\text{def}}{=} \mathcal{M}' \setminus \{1\}$, we have that $\mathcal{S}'$ is isomorphic to the Burnside semigroup.

For convenience, we are going to analyse the structure of the Burnside monoid looking at $\mathcal{M}'$ (and several times we will also consider the semigroup $\mathcal{S}'$). Note that A is a set of generators for $\mathcal{M}'$ and for $\mathcal{S}'$. It can be shown that the relations $\mathcal{J}$ and $\mathcal{D}$ are the same in every Burnside semigroups.

As the definition of $\mathcal{M}'$ supposes that the productions of Σ are stable, and this fact is guaranteed by the Stability Theorem when $n \geq 4$ and $m \geq 1$, we will admit from now on that the productions of Σ are stable. This hipothesis will not even be mentioned explicitly from now on.

7.3 $\mathcal{R}$-entrances, $\mathcal{L}$-entrances e $\mathcal{D}$-entrances

Let $u \in \mathcal{S}'$. We say that u is an *$\mathcal{R}$-entrance* to its $\mathcal{R}$-class when $\exists v \in \mathcal{M}'$ and $\exists a \in A$ such that $v \cdot a = u \not\mathcal{R}\, v$. We say that u is an *$\mathcal{L}$-entrance* to its $\mathcal{L}$-class when $\exists v \in \mathcal{M}'$ and $\exists a \in A$ such that $a \cdot v = u \not\mathcal{L}\, v$. We say that u is a *$\mathcal{D}$-entrance* to its $\mathcal{D}$-class when it is an $\mathcal{R}$-entrance and an $\mathcal{L}$-entrance simultaneously.

Intuitively, $\mathcal{R}$-entrances are the "doors" through which we have to enter if we are outside of an $\mathcal{R}$-class, and we intend to enter it by right multiplication. The $\mathcal{L}$-entrances have a dual interpretation. In the case of a $\mathcal{D}$-class, we allow right and/or left multiplications and every $\mathcal{R}$-entrance or $\mathcal{L}$-entrance would be such a door. We define as $\mathcal{D}$-entrances only the "most important" doors; they are the doors for right and left multiplications, they are "wider" doors.

We also define the sets

$$R \stackrel{\text{def}}{=} \{w \in \mathcal{S}' \text{ such that } |w \rightleftharpoons l_\sigma| \leq |c_\sigma| \text{ for every } \sigma \in \Sigma\}$$

and

$$L \stackrel{\text{def}}{=} \{w \in \mathcal{S}' \text{ such that } |l_\sigma \rightleftharpoons w| \leq |c_\sigma| \text{ for every } \sigma \in \Sigma\}.$$

If $w \in R$ we define the *right handle* of w as the longest suffix of w which is a short of some production of Σ if there is such a suffix and as 1 otherwise. We can define the dual concept of *left handle* of an element of L. We will show, in theorem 7.2, that R is the set of $\mathcal{R}$-entrances and there is a unique $\mathcal{R}$-entrance in an $\mathcal{R}$-class. Dually, L is the set of $\mathcal{L}$-entrances. It follows that $R \cap L$ is the set of $\mathcal{D}$-entrances. Lemma 7.7 will show us that the structure of any $\mathcal{R}$-class depends only on the right handle of its $\mathcal{R}$-entrance.

Proposition 7.1 *Let $u \in \mathcal{M}'$ and let $a \in A$. Then we have that $u \cdot a \neq ua$ if and only if there is $\sigma \in \Sigma$ such that $l_\sigma \in \mathrm{Suf}(ua)$, which, in turn, implies that $u \cdot a$ is the prefix $(ua)\mathrm{b_R}(\sigma)^{-m}$ of u and it is in the same $\mathcal{R}$-class as u.*

The theorem that follows is basic for a more refined analysis of the structure of the $\mathcal{R}$-classes and for other structural properties of $\mathcal{M}'$ that we will see. It tells us that the $\mathcal{R}$-entrance of the $\mathcal{R}$-class of a given element of $\mathcal{S}'$ is unique and it identifies us such an entrance. Intuitively speaking, it is the word obtained by erasing letters of w from the right until we obtain a word in R. By duality, erasing letters of w from the left until we obtain a word in L, we have the $\mathcal{L}$-entrance of $\mathcal{L}$-class of w.

Theorem 7.2 *The following assertions hold.*

1. *Each $\mathcal{R}$-class of $\mathcal{S}'$ has exaclty one $\mathcal{R}$-entrance.*
2. *The set of $\mathcal{R}$-entrances is the set R defined above.*
3. *If w is an $\mathcal{R}$-entrance and $v \in \mathcal{M}'$, we have that $v \leq_{\mathcal{R}} w$ iff w is a prefix of v. In particular, w is a prefix of all elements of its $\mathcal{R}$-class.*
4. *If w is an $\mathcal{R}$-entrance, then there is a unique $u \in \mathcal{M}'$ and a unique $a \in A$ such that $u \cdot a = w \not\mathcal{R}\, u$, and $w = ua$.*

Corollary 7.3 *The frame of the $\mathcal{R}$-classes is a tree.*

Proof. Using items 1 and 4 of theorem 7.2 for a given $\mathcal{R}$-class, we see that there is only one $\mathcal{R}$-class immediatly $\mathcal{R}$-above. □

Corollary 7.4 *The set of $\mathcal{D}$-entrances is $R \cap L$.*

The theorem that follows gives us a characterization of the rigid $\mathcal{R}$-classes. Note that its condition is equivalent to saying that the right handle of w is empty.

Theorem 7.5 *Let $w \in \mathcal{S}'$. Then the $\mathcal{R}$-class of w is rigid iff $w \in R \setminus A^* c_\Sigma$.*

7.4 Regular and irregular $\mathcal{D}$-classes of $\mathcal{S}'$

The following lemma is very important to obtain finiteness properties in the Burnside semigroups in question. Its proof is complex and involves a local analysis of sequences of expansions. Such analysis describes the possible interactions between two consecutive expansions.

Lemma 7.6 *Let $w \in \mathcal{S}'$ be an $\mathcal{R}$-entrance of its $\mathcal{R}$-class. Let v be an element of the $\mathcal{R}$-class of w. Let $y \in \mathcal{M}'$ be a word of minimal length such that $w = v \cdot y$. There is a unique sequence of expansions that expands w to vy and the bases of the involved productions form a strictly decreasing sequence.*

The lemma that follows gives us an information very similar to that given by Green's Lemma (known in the study of semigroups) about $\mathcal{R}$-classes inside the same $\mathcal{D}$-class. The difference is that here we are possibly dealing with bijections between $\mathcal{R}$-classes of different $\mathcal{D}$-classes. We show that the internal structure of an $\mathcal{R}$-class (and in particular its size) is determined exclusively by the right handle of its $\mathcal{R}$-entrance.

Lemma 7.7 *Let w be an $\mathcal{R}$-entrance. Let $u \in c_\Sigma \cup \{1\}$ be the right handle of w and let $v \in \mathrm{Pref}(w)$ be such that $w = vu$. Let $\rho : \mathcal{R}\text{-class of } u \longrightarrow A^*$ be the function defined by $\rho(u') \stackrel{\text{def}}{=} vu'$. Then $\rho(u') = vu' = v \cdot u'$ and the function ρ is a bijection between the $\mathcal{R}$-classes of u and of w. Given u' in the $\mathcal{R}$-class of u and w' in the $\mathcal{R}$-class of w such that $\rho(u') = w'$, for every $x \in \mathcal{M}'$, we have that $u' \cdot x \mathrel{\mathcal{R}} u \Longrightarrow w' \cdot x = \rho(u' \cdot x) \mathrel{\mathcal{R}} w$, and that $w' \cdot x \mathrel{\mathcal{R}} w \Longrightarrow u' \cdot x = \rho^{-1}(w' \cdot x) \mathrel{\mathcal{R}} u$.*

Corollary 7.8 *The internal structure (including its size) of an $\mathcal{R}$-class of $\mathcal{S}'$ is determined exclusively by the right handle of its $\mathcal{R}$-entrance.*

In the following theorem we show that the only regular $\mathcal{D}$-classes are those in which there is one $\mathcal{D}$-entrance that is a short of a production of Σ. We can also tell when two of them are in the same $\mathcal{D}$-class. Note that, from corollary 7.4, we are able to know what are the $\mathcal{D}$-entrances of irregular $\mathcal{D}$-classes then. In this situation, the information that the $\mathcal{D}$-entrances are unique in such $\mathcal{D}$-classes is of capital importance.

Theorem 7.9 *Every short of any production of Σ is a $\mathcal{D}$-entrance of some regular $\mathcal{D}$-class. The $\mathcal{D}$-entrances of a regular $\mathcal{D}$-class of $\mathcal{S}'$ consist of the short of some production σ of Σ and the shorts of all productions of Σ conjugate to σ. An irregular $\mathcal{D}$-class of $\mathcal{S}'$ has a unique $\mathcal{D}$-entrance, which, for its turn, is a factor of all other elements of this $\mathcal{D}$-class.*

Theorem 7.10 *The regular $\mathcal{H}$-classes of $\mathcal{S}'$ have order m and the maximal subgroups of $\mathcal{S}'$ are all cyclic of order m. On the other hand, the irregular $\mathcal{H}$-classes are trivial.*

7.5 The finiteness $\mathcal{J}$-above and the Conjecture of Brzozowski

Given an $\mathcal{R}$-entrance w and given an element v in its $\mathcal{R}$-class, from lemma 7.6, we can associate to v a unique sequence of expansions from w under Σ. Moreover, the bases of expansions form a strictly decreasing sequence. Each of these sequences can be associated to finitely many elements of the $\mathcal{R}$-class (it can be shown that such index of finiteness is, at least, the base of the production used in the last expansion). As it is easy to verify that there are finitely many sequences of expansions of this kind from w — the mere fact that the bases of the productions used in the expansions form a strictly decreasing sequence implies it — the $\mathcal{R}$-classes are finite. Together with corollary 7.3 this fact shows that the semigroup $\mathcal{S}'$ is finite $\mathcal{R}$-above. By duality it is also finite $\mathcal{L}$-above and it is then immediate to verify the following proposition.

Theorem 7.11 *The semigroup $\mathcal{S}'$ is finite $\mathcal{J}$-above.*

Corollary 7.12 *The conjecture of Brzozowski given in the introduction is true whenever all productions of Σ are stable. In particular it holds for the cases in which $n \geq 4$ and $m \geq 1$.*

Proof. Let $w' \in A^*$ and let $w = \mathrm{rep}(w')$. Then $I = \mathcal{M}' \setminus \mathrm{Up}_{\mathcal{J}}(w)$ is an ideal. Let S be the semigroup $\mathcal{M}'/I$, that is, the semigroup obtained from $\mathcal{M}'$ by the contraction of I to a single element, namely 0, which is a zero of $\mathcal{M}'/I$. This semigroup $\mathcal{M}'/I$ is finite because of theorem 7.11. Let $f : A^* \longrightarrow \mathcal{M}'/I$ be the function defined by $f(u) = \mathrm{rep}(u)$ if $\mathrm{rep}(u) \in \mathrm{Up}_{\mathcal{J}}(w)$ and $f(u) = 0$ otherwise. This function is an epimorphism and $f(w') = \mathrm{rep}(w') = w$ and $f^{-1}f(w') = [w] = [w']$. Thus, $[w']$ is recognizable. □

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