## Definition of Vector Space

Before defining what a vector space is, let's look at two important examples. The vector space  $\mathbf{R}^2$ , which you can think of as a plane, consists of all ordered pairs of real numbers:

$$\mathbf{R}^2 = \{(x, y) : x, y \in \mathbf{R}\}.$$

The vector space  $\mathbf{R}^3$ , which you can think of as ordinary space, consists of all ordered triples of real numbers:

$$\mathbf{R}^3 = \{(x, y, z) : x, y, z \in \mathbf{R}\}.$$

To generalize  $\mathbf{R}^2$  and  $\mathbf{R}^3$  to higher dimensions, we first need to discuss the concept of lists. Suppose *n* is a nonnegative integer. A *list* of *length n* is an ordered collection of *n* objects (which might be numbers, other lists, or more abstract entities) separated by commas and surrounded by parentheses. A list of length *n* looks like this:

Many mathematicians call a list of length n an n-**tuple**.

 $(x_1,\ldots,x_n).$ 

Thus a list of length 2 is an ordered pair and a list of length 3 is an ordered triple. For  $j \in \{1, ..., n\}$ , we say that  $x_j$  is the j<sup>th</sup> *coordinate* of the list above. Thus  $x_1$  is called the first coordinate,  $x_2$  is called the second coordinate, and so on.

Sometimes we will use the word *list* without specifying its length. Remember, however, that by definition each list has a finite length that is a nonnegative integer, so that an object that looks like

$$(x_1, x_2, ...),$$

which might be said to have infinite length, is not a list. A list of length 0 looks like this: (). We consider such an object to be a list so that some of our theorems will not have trivial exceptions.

Two lists are equal if and only if they have the same length and the same coordinates in the same order. In other words,  $(x_1, ..., x_m)$  equals  $(y_1, ..., y_n)$  if and only if m = n and  $x_1 = y_1, ..., x_m = y_m$ .

Lists differ from sets in two ways: in lists, order matters and repetitions are allowed, whereas in sets, order and repetitions are irrelevant. For example, the lists (3, 5) and (5, 3) are not equal, but the sets  $\{3, 5\}$ and  $\{5, 3\}$  are equal. The lists (4, 4) and (4, 4, 4) are not equal (they do not have the same length), though the sets  $\{4,4\}$  and  $\{4,4,4\}$  both equal the set  $\{4\}$ .

To define the higher-dimensional analogues of  $\mathbf{R}^2$  and  $\mathbf{R}^3$ , we will simply replace  $\mathbf{R}$  with  $\mathbf{F}$  (which equals  $\mathbf{R}$  or  $\mathbf{C}$ ) and replace the 2 or 3 with an arbitrary positive integer. Specifically, fix a positive integer nfor the rest of this section. We define  $\mathbf{F}^n$  to be the set of all lists of length n consisting of elements of  $\mathbf{F}$ :

$$\mathbf{F}^n = \{ (x_1, \dots, x_n) : x_j \in \mathbf{F} \text{ for } j = 1, \dots, n \}.$$

For example, if  $\mathbf{F} = \mathbf{R}$  and *n* equals 2 or 3, then this definition of  $\mathbf{F}^n$  agrees with our previous notions of  $\mathbf{R}^2$  and  $\mathbf{R}^3$ . As another example,  $\mathbf{C}^4$  is the set of all lists of four complex numbers:

$$\mathbf{C}^4 = \{(z_1, z_2, z_3, z_4) : z_1, z_2, z_3, z_4 \in \mathbf{C}\}.$$

If  $n \ge 4$ , we cannot easily visualize  $\mathbf{R}^n$  as a physical object. The same problem arises if we work with complex numbers:  $\mathbf{C}^1$  can be thought of as a plane, but for  $n \ge 2$ , the human brain cannot provide geometric models of  $\mathbf{C}^n$ . However, even if n is large, we can perform algebraic manipulations in  $\mathbf{F}^n$  as easily as in  $\mathbf{R}^2$  or  $\mathbf{R}^3$ . For example, addition is defined on  $\mathbf{F}^n$  by adding corresponding coordinates:

1.1 
$$(x_1,...,x_n) + (y_1,...,y_n) = (x_1 + y_1,...,x_n + y_n).$$

Often the mathematics of  $\mathbf{F}^n$  becomes cleaner if we use a single entity to denote an list of *n* numbers, without explicitly writing the coordinates. Thus the commutative property of addition on  $\mathbf{F}^n$  should be expressed as

$$x + y = y + x$$

for all  $x, y \in \mathbf{F}^n$ , rather than the more cumbersome

$$(x_1, \ldots, x_n) + (y_1, \ldots, y_n) = (y_1, \ldots, y_n) + (x_1, \ldots, x_n)$$

for all  $x_1, ..., x_n, y_1, ..., y_n \in \mathbf{F}$  (even though the latter formulation is needed to prove commutativity). If a single letter is used to denote an element of  $\mathbf{F}^n$ , then the same letter, with appropriate subscripts, is often used when coordinates must be displayed. For example, if  $x \in \mathbf{F}^n$ , then letting x equal  $(x_1, ..., x_n)$  is good notation. Even better, work with just x and avoid explicit coordinates, if possible. For an amusing account of how  $\mathbb{R}^3$ would be perceived by a creature living in  $\mathbb{R}^2$ , read Flatland: A Romance of Many Dimensions, by Edwin A. Abbott. This novel, published in 1884, can help creatures living in three-dimensional space, such as ourselves, imagine a physical space of four or more dimensions. We let 0 denote the list of length *n* all of whose coordinates are 0:

$$0=(0,\ldots,0)$$

Note that we are using the symbol 0 in two different ways—on the left side of the equation above, 0 denotes a list of length n, whereas on the right side, each 0 denotes a number. This potentially confusing practice actually causes no problems because the context always makes clear what is intended. For example, consider the statement that 0 is an additive identity for  $\mathbf{F}^n$ :

x + 0 = x

for all  $x \in \mathbf{F}^n$ . Here 0 must be a list because we have not defined the sum of an element of  $\mathbf{F}^n$  (namely, x) and the number 0.

A picture can often aid our intuition. We will draw pictures depicting  $\mathbf{R}^2$  because we can easily sketch this space on two-dimensional surfaces such as paper and blackboards. A typical element of  $\mathbf{R}^2$  is a point  $x = (x_1, x_2)$ . Sometimes we think of x not as a point but as an arrow starting at the origin and ending at  $(x_1, x_2)$ , as in the picture below. When we think of x as an arrow, we refer to it as a *vector*.



Elements of  $\mathbf{R}^2$  can be thought of as points or as vectors.

The coordinate axes and the explicit coordinates unnecessarily clutter the picture above, and often you will gain better understanding by dispensing with them and just thinking of the vector, as in the next picture.



Whenever we use pictures in  $\mathbf{R}^2$  or use the somewhat vague language of points and vectors, remember that these are just aids to our understanding, not substitutes for the actual mathematics that we will develop. Though we cannot draw good pictures in high-dimensional spaces, the elements of these spaces are as rigorously defined as elements of  $\mathbf{R}^2$ . For example,  $(2, -3, 17, \pi, \sqrt{2})$  is an element of  $\mathbf{R}^5$ , and we may casually refer to it as a point in  $\mathbf{R}^5$  or a vector in  $\mathbf{R}^5$  without worrying about whether the geometry of  $\mathbf{R}^5$  has any physical meaning.

Recall that we defined the sum of two elements of  $\mathbf{F}^n$  to be the element of  $\mathbf{F}^n$  obtained by adding corresponding coordinates; see 1.1. In the special case of  $\mathbf{R}^2$ , addition has a simple geometric interpretation. Suppose we have two vectors x and y in  $\mathbf{R}^2$  that we want to add, as in the left side of the picture below. Move the vector y parallel to itself so that its initial point coincides with the end point of the vector x. The sum x + y then equals the vector whose initial point equals the initial point of x and whose end point equals the end point of the moved vector y, as in the right side of the picture below.



Mathematical models of the economy often have thousands of variables, say  $x_1, \ldots, x_{5000}$ , which means that we must operate in  $\mathbf{R}^{5000}$ . Such a space cannot be dealt with geometrically, but the algebraic approach works well. That's why our subject is called linear **algebra**.

Our treatment of the vector y in the picture above illustrates a standard philosophy when we think of vectors in  $\mathbf{R}^2$  as arrows: we can move an arrow parallel to itself (not changing its length or direction) and still think of it as the same vector.

Having dealt with addition in  $\mathbf{F}^n$ , we now turn to multiplication. We could define a multiplication on  $\mathbf{F}^n$  in a similar fashion, starting with two elements of  $\mathbf{F}^n$  and getting another element of  $\mathbf{F}^n$  by multiplying corresponding coordinates. Experience shows that this definition is not useful for our purposes. Another type of multiplication, called scalar multiplication, will be central to our subject. Specifically, we need to define what it means to multiply an element of  $\mathbf{F}^n$  by an element of  $\mathbf{F}$ . We make the obvious definition, performing the multiplication in each coordinate:

$$a(x_1,\ldots,x_n)=(ax_1,\ldots,ax_n);$$

here  $a \in \mathbf{F}$  and  $(x_1, \ldots, x_n) \in \mathbf{F}^n$ .

Scalar multiplication has a nice geometric interpretation in  $\mathbb{R}^2$ . If *a* is a positive number and *x* is a vector in  $\mathbb{R}^2$ , then *ax* is the vector that points in the same direction as *x* and whose length is *a* times the length of *x*. In other words, to get *ax*, we shrink or stretch *x* by a factor of *a*, depending upon whether *a* < 1 or *a* > 1. The next picture illustrates this point.



If *a* is a negative number and *x* is a vector in  $\mathbb{R}^2$ , then *ax* is the vector that points in the opposite direction as *x* and whose length is |a| times the length of *x*, as illustrated in the next picture.



Multiplication by negative scalars

In scalar multiplication, we multiply together a scalar and a vector, getting a vector. You may be familiar with the dot product in  $\mathbf{R}^2$ or  $\mathbf{R}^3$ , in which we multiply together two vectors and obtain a scalar. Generalizations of the dot product will become important when we study inner products in Chapter 6. You may also be familiar with the cross product in  $\mathbf{R}^3$ , in which we multiply together two vectors and obtain another vector. No useful generalization of this type of multiplication exists in higher dimensions.

The motivation for the definition of a vector space comes from the important properties possessed by addition and scalar multiplication on  $\mathbf{F}^n$ . Specifically, addition on  $\mathbf{F}^n$  is commutative and associative and has an identity, namely, 0. Every element has an additive inverse. Scalar multiplication on  $\mathbf{F}^n$  is associative, and scalar multiplication by 1 acts as a multiplicative identity should. Finally, addition and scalar multiplication on  $\mathbf{F}^n$  are connected by distributive properties.

We will define a vector space to be a set *V* along with an addition and a scalar multiplication on *V* that satisfy the properties discussed in the previous paragraph. By an *addition* on *V* we mean a function that assigns an element  $u + v \in V$  to each pair of elements  $u, v \in V$ . By a *scalar multiplication* on *V* we mean a function that assigns an element  $av \in V$  to each  $a \in \mathbf{F}$  and each  $v \in V$ .

Now we are ready to give the formal definition of a vector space. A *vector space* is a set V along with an addition on V and a scalar multiplication on V such that the following properties hold:

## commutativity

u + v = v + u for all  $u, v \in V$ ;

#### associativity

(u + v) + w = u + (v + w) and (ab)v = a(bv) for all  $u, v, w \in V$ and all  $a, b \in \mathbf{F}$ ;

## additive identity

there exists an element  $0 \in V$  such that  $\nu + 0 = \nu$  for all  $\nu \in V$ ;

#### additive inverse

for every  $\nu \in V$ , there exists  $w \in V$  such that  $\nu + w = 0$ ;

## multiplicative identity

 $1\nu = \nu$  for all  $\nu \in V$ ;

## distributive properties

a(u + v) = au + av and (a + b)u = au + bu for all  $a, b \in F$  and all  $u, v \in V$ .

The scalar multiplication in a vector space depends upon **F**. Thus when we need to be precise, we will say that *V* is a vector space over **F** instead of saying simply that *V* is a vector space. For example,  $\mathbf{R}^n$  is a vector space over **R**, and  $\mathbf{C}^n$  is a vector space over **C**. Frequently, a vector space over **R** is called a *real vector space* and a vector space over

C is called a *complex vector space*. Usually the choice of F is either obvious from the context or irrelevant, and thus we often assume that F is lurking in the background without specifically mentioning it.

Elements of a vector space are called *vectors* or *points*. This geometric language sometimes aids our intuition.

Not surprisingly,  $\mathbf{F}^n$  is a vector space over  $\mathbf{F}$ , as you should verify. Of course, this example motivated our definition of vector space.

For another example, consider  $F^{\infty}$ , which is defined to be the set of all sequences of elements of F:

$$\mathbf{F}^{\infty} = \{(x_1, x_2, \dots) : x_j \in \mathbf{F} \text{ for } j = 1, 2, \dots\}.$$

Addition and scalar multiplication on  $\mathbf{F}^{\infty}$  are defined as expected:

$$(x_1, x_2, \dots) + (y_1, y_2, \dots) = (x_1 + y_1, x_2 + y_2, \dots)$$
  
 $a(x_1, x_2, \dots) = (ax_1, ax_2, \dots).$ 

With these definitions,  $F^{\infty}$  becomes a vector space over F, as you should verify. The additive identity in this vector space is the sequence consisting of all 0's.

Our next example of a vector space involves polynomials. A function  $p: \mathbf{F} \rightarrow \mathbf{F}$  is called a *polynomial* with coefficients in  $\mathbf{F}$  if there exist  $a_0, \ldots, a_m \in \mathbf{F}$  such that

$$p(z) = a_0 + a_1 z + a_2 z^2 + \dots + a_m z^m$$

for all  $z \in \mathbf{F}$ . We define  $\mathcal{P}(\mathbf{F})$  to be the set of all polynomials with coefficients in  $\mathbf{F}$ . Addition on  $\mathcal{P}(\mathbf{F})$  is defined as you would expect: if  $p, q \in \mathcal{P}(\mathbf{F})$ , then p + q is the polynomial defined by

$$(p+q)(z) = p(z) + q(z)$$

for  $z \in \mathbf{F}$ . For example, if p is the polynomial defined by  $p(z) = 2z + z^3$ and q is the polynomial defined by q(z) = 7 + 4z, then p + q is the polynomial defined by  $(p + q)(z) = 7 + 6z + z^3$ . Scalar multiplication on  $\mathcal{P}(\mathbf{F})$  also has the obvious definition: if  $a \in \mathbf{F}$  and  $p \in \mathcal{P}(\mathbf{F})$ , then ap is the polynomial defined by

$$ap(z) = ap(z)$$

for  $z \in \mathbf{F}$ . With these definitions of addition and scalar multiplication,  $\mathcal{P}(\mathbf{F})$  is a vector space, as you should verify. The additive identity in this vector space is the polynomial all of whose coefficients equal 0.

Soon we will see further examples of vector spaces, but first we need to develop some of the elementary properties of vector spaces.

The simplest vector space contains only one point. In other words, {0} is a vector space, though not a very interesting one.

Though F<sup>n</sup> is our crucial example of a vector space, not all vector spaces consist of lists. For example, the elements of  $\mathcal{P}(\mathbf{F})$ consist of functions on F, not lists. In general, a vector space is an abstract entity whose elements might be lists, functions, or weird objects.

## Properties of Vector Spaces

The definition of a vector space requires that it have an additive identity. The proposition below states that this identity is unique.

**1.2 Proposition:** A vector space has a unique additive identity.

**PROOF:** Suppose 0 and 0' are both additive identities for some vector space V. Then

$$0' = 0' + 0 = 0,$$

where the first equality holds because 0 is an additive identity and the second equality holds because 0' is an additive identity. Thus 0' = 0, proving that *V* has only one additive identity.

Each element  $\nu$  in a vector space has an additive inverse, an element w in the vector space such that  $\nu + w = 0$ . The next proposition shows that each element in a vector space has only one additive inverse.

**1.3 Proposition:** *Every element in a vector space has a unique additive inverse.* 

**PROOF:** Suppose *V* is a vector space. Let  $v \in V$ . Suppose that *w* and *w*' are additive inverses of *v*. Then

w = w + 0 = w + (v + w') = (w + v) + w' = 0 + w' = w'.

Thus w = w', as desired.

Because additive inverses are unique, we can let  $-\nu$  denote the additive inverse of a vector  $\nu$ . We define  $w - \nu$  to mean  $w + (-\nu)$ .

Almost all the results in this book will involve some vector space. To avoid being distracted by having to restate frequently something such as "Assume that V is a vector space", we now make the necessary declaration once and for all:

Let's agree that for the rest of the book *V* will denote a vector space over **F**.

The symbol ∎ means "end of the proof".

Because of associativity, we can dispense with parentheses when dealing with additions involving more than two elements in a vector space. For example, we can write u + v + w without parentheses because the two possible interpretations of that expression, namely, (u + v) + w and u + (v + w), are equal. We first use this familiar convention of not using parentheses in the next proof. In the next proposition, 0 denotes a scalar (the number  $0 \in \mathbf{F}$ ) on the left side of the equation and a vector (the additive identity of *V*) on the right side of the equation.

**1.4 Proposition:** 0v = 0 for every  $v \in V$ .

**PROOF:** For  $\nu \in V$ , we have

$$0\nu = (0+0)\nu = 0\nu + 0\nu.$$

Adding the additive inverse of  $0\nu$  to both sides of the equation above gives  $0 = 0\nu$ , as desired.

In the next proposition, 0 denotes the additive identity of V. Though their proofs are similar, 1.4 and 1.5 are not identical. More precisely, 1.4 states that the product of the scalar 0 and any vector equals the vector 0, whereas 1.5 states that the product of any scalar and the vector 0 equals the vector 0.

**1.5 Proposition:** a0 = 0 for every  $a \in \mathbf{F}$ .

**PROOF:** For  $a \in \mathbf{F}$ , we have

a0 = a(0+0) = a0 + a0.

Adding the additive inverse of a0 to both sides of the equation above gives 0 = a0, as desired.

Now we show that if an element of *V* is multiplied by the scalar -1, then the result is the additive inverse of the element of *V*.

#### **1.6** Proposition: (-1)v = -v for every $v \in V$ .

**PROOF:** For  $\nu \in V$ , we have

 $\nu + (-1)\nu = 1\nu + (-1)\nu = (1 + (-1))\nu = 0\nu = 0.$ 

This equation says that  $(-1)\nu$ , when added to  $\nu$ , gives 0. Thus  $(-1)\nu$  must be the additive inverse of  $\nu$ , as desired.

Note that 1.4 and 1.5 assert something about scalar multiplication and the additive identity of V. The only part of the definition of a vector space that connects scalar multiplication and vector addition is the distributive property. Thus the distributive property must be used in the proofs.

## Subspaces

A subset U of V is called a *subspace* of V if U is also a vector space (using the same addition and scalar multiplication as on V). For example,

$$\{(x_1, x_2, 0) : x_1, x_2 \in \mathbf{F}\}$$

is a subspace of **F**<sup>3</sup>.

If *U* is a subset of *V*, then to check that *U* is a subspace of *V* we need only check that U satisfies the following:

#### additive identity

 $0 \in U$ 

## closed under addition

 $u, v \in U$  implies  $u + v \in U$ ;

## closed under scalar multiplication

 $a \in \mathbf{F}$  and  $u \in U$  implies  $au \in U$ .

Some mathematicians use the term linear subspace, which means the same as subspace.

Clearly {0} is the smallest subspace of V and V itself is the largest subspace of V. The empty set is not a subspace of V because a subspace must be a vector space and a vector space must contain at least one element, namely, an additive identity.

The first condition insures that the additive identity of V is in U. The second condition insures that addition makes sense on U. The third condition insures that scalar multiplication makes sense on U. To show that *U* is a vector space, the other parts of the definition of a vector space do not need to be checked because they are automatically satisfied. For example, the associative and commutative properties of addition automatically hold on U because they hold on the larger space V. As another example, if the third condition above holds and  $u \in U$ , then -u (which equals (-1)u by 1.6) is also in *U*, and hence every element of U has an additive inverse in U.

The three conditions above usually enable us to determine quickly whether a given subset of *V* is a subspace of *V*. For example, if  $b \in \mathbf{F}$ , then

$$\{(x_1, x_2, x_3, x_4) \in \mathbf{F}^4 : x_3 = 5x_4 + b\}$$

is a subspace of  $\mathbf{F}^4$  if and only if b = 0, as you should verify. As another example, you should verify that

$$\{p \in \mathcal{P}(\mathbf{F}) : p(3) = 0\}$$

is a subspace of  $\mathcal{P}(\mathbf{F})$ .

The subspaces of  $\mathbf{R}^2$  are precisely {0},  $\mathbf{R}^2$ , and all lines in  $\mathbf{R}^2$  through the origin. The subspaces of  $\mathbf{R}^3$  are precisely {0},  $\mathbf{R}^3$ , all lines in  $\mathbf{R}^3$ 

through the origin, and all planes in  $\mathbb{R}^3$  through the origin. To prove that all these objects are indeed subspaces is easy—the hard part is to show that they are the only subspaces of  $\mathbb{R}^2$  or  $\mathbb{R}^3$ . That task will be easier after we introduce some additional tools in the next chapter.

## Sums and Direct Sums

In later chapters, we will find that the notions of vector space sums and direct sums are useful. We define these concepts here.

Suppose  $U_1, \ldots, U_m$  are subspaces of V. The *sum* of  $U_1, \ldots, U_m$ , denoted  $U_1 + \cdots + U_m$ , is defined to be the set of all possible sums of elements of  $U_1, \ldots, U_m$ . More precisely,

$$U_1 + \cdots + U_m = \{u_1 + \cdots + u_m : u_1 \in U_1, \dots, u_m \in U_m\}$$

You should verify that if  $U_1, \ldots, U_m$  are subspaces of V, then the sum  $U_1 + \cdots + U_m$  is a subspace of V.

Let's look at some examples of sums of subspaces. Suppose *U* is the set of all elements of  $\mathbf{F}^3$  whose second and third coordinates equal 0, and *W* is the set of all elements of  $\mathbf{F}^3$  whose first and third coordinates equal 0:

$$U = \{(x, 0, 0) \in \mathbf{F}^3 : x \in \mathbf{F}\}$$
 and  $W = \{(0, y, 0) \in \mathbf{F}^3 : y \in \mathbf{F}\}.$ 

Then

1.7

$$U + W = \{(x, y, 0) : x, y \in \mathbf{F}\}$$

*to* as you should verify. *Set* As another examp

As another example, suppose U is as above and W is the set of all elements of  $\mathbf{F}^3$  whose first and second coordinates equal each other and whose third coordinate equals 0:

$$W = \{(\gamma, \gamma, 0) \in \mathbf{F}^3 : \gamma \in \mathbf{F}\}.$$

Then U + W is also given by 1.7, as you should verify.

Suppose  $U_1, \ldots, U_m$  are subspaces of *V*. Clearly  $U_1, \ldots, U_m$  are all contained in  $U_1 + \cdots + U_m$  (to see this, consider sums  $u_1 + \cdots + u_m$  where all except one of the *u*'s are 0). Conversely, any subspace of *V* containing  $U_1, \ldots, U_m$  must contain  $U_1 + \cdots + U_m$  (because subspaces

When dealing with vector spaces, we are usually interested only in subspaces, as opposed to arbitrary subsets. The union of subspaces is rarely a subspace (see Exercise 9 in this chapter), which is why we usually work with sums rather than unions.

Sums of subspaces in the theory of vector spaces are analogous to unions of subsets in set theory. Given two subspaces of a vector space, the smallest subspace containing them is their sum. Analogously, given two subsets of a set, the smallest subset containing them is their union. must contain all finite sums of their elements). Thus  $U_1 + \cdots + U_m$  is the smallest subspace of *V* containing  $U_1, \ldots, U_m$ .

Suppose  $U_1, \ldots, U_m$  are subspaces of *V* such that  $V = U_1 + \cdots + U_m$ . Thus every element of *V* can be written in the form

$$u_1 + \cdots + u_m$$
,

where each  $u_j \in U_j$ . We will be especially interested in cases where each vector in *V* can be uniquely represented in the form above. This situation is so important that we give it a special name: direct sum. Specifically, we say that *V* is the *direct sum* of subspaces  $U_1, \ldots, U_m$ , written  $V = U_1 \oplus \cdots \oplus U_m$ , if each element of *V* can be written uniquely as a sum  $u_1 + \cdots + u_m$ , where each  $u_j \in U_j$ .

Let's look at some examples of direct sums. Suppose *U* is the subspace of  $\mathbf{F}^3$  consisting of those vectors whose last coordinate equals 0, and *W* is the subspace of  $\mathbf{F}^3$  consisting of those vectors whose first two coordinates equal 0:

$$U = \{(x, y, 0) \in \mathbf{F}^3 : x, y \in \mathbf{F}\}$$
 and  $W = \{(0, 0, z) \in \mathbf{F}^3 : z \in \mathbf{F}\}.$ 

Then  $\mathbf{F}^3 = U \oplus W$ , as you should verify.

As another example, suppose  $U_j$  is the subspace of  $\mathbf{F}^n$  consisting of those vectors whose coordinates are all 0, except possibly in the  $j^{\text{th}}$  slot (for example,  $U_2 = \{(0, x, 0, \dots, 0) \in \mathbf{F}^n : x \in \mathbf{F}\}$ ). Then

$$\mathbf{F}^n = U_1 \oplus \cdots \oplus U_n,$$

as you should verify.

As a final example, consider the vector space  $\mathcal{P}(\mathbf{F})$  of all polynomials with coefficients in **F**. Let  $U_e$  denote the subspace of  $\mathcal{P}(\mathbf{F})$  consisting of all polynomials p of the form

$$p(z) = a_0 + a_2 z^2 + \cdots + a_{2m} z^{2m},$$

and let  $U_o$  denote the subspace of  $\mathcal{P}(\mathbf{F})$  consisting of all polynomials p of the form

$$p(z) = a_1 z + a_3 z^3 + \cdots + a_{2m+1} z^{2m+1};$$

here *m* is a nonnegative integer and  $a_0, ..., a_{2m+1} \in \mathbf{F}$  (the notations  $U_e$  and  $U_o$  should remind you of even and odd powers of *z*). You should verify that

The symbol  $\oplus$ , consisting of a plus sign inside a circle, is used to denote direct sums as a reminder that we are dealing with a special type of sum of subspaces—each element in the direct sum can be represented only one way as a sum of elements from the specified subspaces.

$$\mathcal{P}(\mathbf{F}) = U_e \oplus U_o.$$

Sometimes nonexamples add to our understanding as much as examples. Consider the following three subspaces of  $F^3$ :

$$U_1 = \{ (x, y, 0) \in \mathbf{F}^3 : x, y \in \mathbf{F} \};$$
  

$$U_2 = \{ (0, 0, z) \in \mathbf{F}^3 : z \in \mathbf{F} \};$$
  

$$U_3 = \{ (0, y, y) \in \mathbf{F}^3 : y \in \mathbf{F} \}.$$

Clearly  $\mathbf{F}^3 = U_1 + U_2 + U_3$  because an arbitrary vector  $(x, y, z) \in \mathbf{F}^3$  can be written as

$$(x, y, z) = (x, y, 0) + (0, 0, z) + (0, 0, 0),$$

where the first vector on the right side is in  $U_1$ , the second vector is in  $U_2$ , and the third vector is in  $U_3$ . However,  $\mathbf{F}^3$  does not equal the direct sum of  $U_1, U_2, U_3$  because the vector (0, 0, 0) can be written in two different ways as a sum  $u_1+u_2+u_3$ , with each  $u_j \in U_j$ . Specifically, we have

$$(0,0,0) = (0,1,0) + (0,0,1) + (0,-1,-1)$$

and, of course,

$$(0,0,0) = (0,0,0) + (0,0,0) + (0,0,0),$$

where the first vector on the right side of each equation above is in  $U_1$ , the second vector is in  $U_2$ , and the third vector is in  $U_3$ .

In the example above, we showed that something is not a direct sum by showing that 0 does not have a unique representation as a sum of appropriate vectors. The definition of direct sum requires that every vector in the space have a unique representation as an appropriate sum. Suppose we have a collection of subspaces whose sum equals the whole space. The next proposition shows that when deciding whether this collection of subspaces is a direct sum, we need only consider whether 0 can be uniquely written as an appropriate sum.

**1.8 Proposition:** Suppose that  $U_1, \ldots, U_n$  are subspaces of *V*. Then  $V = U_1 \oplus \cdots \oplus U_n$  if and only if both the following conditions hold:

- (a)  $V = U_1 + \cdots + U_n$ ;
- (b) the only way to write 0 as a sum  $u_1 + \cdots + u_n$ , where each  $u_i \in U_i$ , is by taking all the  $u_i$ 's equal to 0.

**PROOF:** First suppose that  $V = U_1 \oplus \cdots \oplus U_n$ . Clearly (a) holds (because of how sum and direct sum are defined). To prove (b), suppose that  $u_1 \in U_1, \ldots, u_n \in U_n$  and

$$0=u_1+\cdots+u_n.$$

Then each  $u_j$  must be 0 (this follows from the uniqueness part of the definition of direct sum because  $0 = 0 + \cdots + 0$  and  $0 \in U_1, \ldots, 0 \in U_n$ ), proving (b).

Now suppose that (a) and (b) hold. Let  $\nu \in V$ . By (a), we can write

$$\nu = u_1 + \cdots + u_n$$

for some  $u_1 \in U_1, ..., u_n \in U_n$ . To show that this representation is unique, suppose that we also have

$$\nu = \nu_1 + \cdots + \nu_n,$$

where  $v_1 \in U_1, \ldots, v_n \in U_n$ . Subtracting these two equations, we have

$$0 = (u_1 - v_1) + \cdots + (u_n - v_n).$$

Clearly  $u_1 - v_1 \in U_1, ..., u_n - v_n \in U_n$ , so the equation above and (b) imply that each  $u_i - v_i = 0$ . Thus  $u_1 = v_1, ..., u_n = v_n$ , as desired.

The next proposition gives a simple condition for testing which pairs of subspaces give a direct sum. Note that this proposition deals only with the case of two subspaces. When asking about a possible direct sum with more than two subspaces, it is not enough to test that any two of the subspaces intersect only at 0. To see this, consider the nonexample presented just before 1.8. In that nonexample, we had  $\mathbf{F}^3 = U_1 + U_2 + U_3$ , but  $\mathbf{F}^3$  did not equal the direct sum of  $U_1, U_2, U_3$ . However, in that nonexample, we have  $U_1 \cap U_2 = U_1 \cap U_3 = U_2 \cap U_3 = \{0\}$ (as you should verify). The next proposition shows that with just two subspaces we get a nice necessary and sufficient condition for a direct sum.

**1.9 Proposition:** *Suppose that U and W are subspaces of V. Then*  $V = U \oplus W$  *if and only if* V = U + W *and*  $U \cap W = \{0\}$ *.* 

**PROOF:** First suppose that  $V = U \oplus W$ . Then V = U + W (by the definition of direct sum). Also, if  $v \in U \cap W$ , then 0 = v + (-v), where

Sums of subspaces are analogous to unions of subsets. Similarly, direct sums of subspaces are analogous to disjoint unions of subsets. No two subspaces of a vector space can be disjoint because both must contain 0. So disjointness is replaced, at least in the case of two subspaces, with the requirement that the intersection equals {0}.

 $\nu \in U$  and  $-\nu \in W$ . By the unique representation of 0 as the sum of a vector in *U* and a vector in *W*, we must have  $\nu = 0$ . Thus  $U \cap W = \{0\}$ , completing the proof in one direction.

To prove the other direction, now suppose that V = U + W and  $U \cap W = \{0\}$ . To prove that  $V = U \oplus W$ , suppose that

$$0 = u + w$$
,

where  $u \in U$  and  $w \in W$ . To complete the proof, we need only show that u = w = 0 (by 1.8). The equation above implies that  $u = -w \in W$ . Thus  $u \in U \cap W$ , and hence u = 0. This, along with equation above, implies that w = 0, completing the proof.

## Exercíses

1. Suppose *a* and *b* are real numbers, not both 0. Find real numbers *c* and *d* such that

1/(a+bi) = c + di.

2. Show that

$$\frac{-1 + \sqrt{3}i}{2}$$

is a cube root of 1 (meaning that its cube equals 1).

- 3. Prove that  $-(-\nu) = \nu$  for every  $\nu \in V$ .
- 4. Prove that if  $a \in \mathbf{F}$ ,  $v \in V$ , and av = 0, then a = 0 or v = 0.
- 5. For each of the following subsets of  $F^3$ , determine whether it is a subspace of  $F^3$ :
  - (a)  $\{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1 + 2x_2 + 3x_3 = 0\};$
  - (b)  $\{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1 + 2x_2 + 3x_3 = 4\};$
  - (c)  $\{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1 x_2 x_3 = 0\};$
  - (d)  $\{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1 = 5x_3\}.$
- 6. Give an example of a nonempty subset U of  $\mathbb{R}^2$  such that U is closed under addition and under taking additive inverses (meaning  $-u \in U$  whenever  $u \in U$ ), but U is not a subspace of  $\mathbb{R}^2$ .
- 7. Give an example of a nonempty subset U of  $\mathbf{R}^2$  such that U is closed under scalar multiplication, but U is not a subspace of  $\mathbf{R}^2$ .
- 8. Prove that the intersection of any collection of subspaces of *V* is a subspace of *V*.
- 9. Prove that the union of two subspaces of *V* is a subspace of *V* if and only if one of the subspaces is contained in the other.
- 10. Suppose that *U* is a subspace of *V*. What is U + U?
- 11. Is the operation of addition on the subspaces of *V* commutative? Associative? (In other words, if  $U_1, U_2, U_3$  are subspaces of *V*, is  $U_1 + U_2 = U_2 + U_1$ ? Is  $(U_1 + U_2) + U_3 = U_1 + (U_2 + U_3)$ ?)

- 12. Does the operation of addition on the subspaces of *V* have an additive identity? Which subspaces have additive inverses?
- 13. Prove or give a counterexample: if  $U_1, U_2, W$  are subspaces of V such that

$$U_1 + W = U_2 + W,$$

then  $U_1 = U_2$ .

14. Suppose *U* is the subspace of  $\mathcal{P}(\mathbf{F})$  consisting of all polynomials *p* of the form

$$p(z) = az^2 + bz^5,$$

where  $a, b \in \mathbf{F}$ . Find a subspace W of  $\mathcal{P}(\mathbf{F})$  such that  $\mathcal{P}(\mathbf{F}) = U \oplus W$ .

15. Prove or give a counterexample: if  $U_1, U_2, W$  are subspaces of V such that

 $V = U_1 \oplus W$  and  $V = U_2 \oplus W$ ,

then  $U_1 = U_2$ .

## Chapter 2

# Fíníte-Dímensíonal Vector Spaces

In the last chapter we learned about vector spaces. Linear algebra focuses not on arbitrary vector spaces, but on finite-dimensional vector spaces, which we introduce in this chapter. Here we will deal with the key concepts associated with these spaces: span, linear independence, basis, and dimension.

Let's review our standing assumptions:

Recall that **F** denotes **R** or **C**. Recall also that V is a vector space over **F**.



## Span and Linear Independence

A *linear combination* of a list  $(v_1, ..., v_m)$  of vectors in *V* is a vector of the form

$$a_1v_1+\cdots+a_mv_m,$$

where  $a_1, \ldots, a_m \in \mathbf{F}$ . The set of all linear combinations of  $(\nu_1, \ldots, \nu_m)$  is called the *span* of  $(\nu_1, \ldots, \nu_m)$ , denoted span $(\nu_1, \ldots, \nu_m)$ . In other words,

span, which means the span $(\nu_1, \dots, \nu_m) = \{a_1\nu_1 + \dots + a_m\nu_m : a_1, \dots, a_m \in \mathbf{F}\}.$ 

As an example of these concepts, suppose  $V = F^3$ . The vector (7, 2, 9) is a linear combination of ((2, 1, 3), (1, 0, 1)) because

$$(7,2,9) = 2(2,1,3) + 3(1,0,1).$$

Thus  $(7, 2, 9) \in \text{span}((2, 1, 3), (1, 0, 1)).$ 

You should verify that the span of any list of vectors in V is a subspace of V. To be consistent, we declare that the span of the empty list () equals {0} (recall that the empty set is not a subspace of V).

If  $(v_1, \ldots, v_m)$  is a list of vectors in *V*, then each  $v_j$  is a linear combination of  $(v_1, \ldots, v_m)$  (to show this, set  $a_j = 1$  and let the other *a*'s in 2.1 equal 0). Thus span $(v_1, \ldots, v_m)$  contains each  $v_j$ . Conversely, because subspaces are closed under scalar multiplication and addition, every subspace of *V* containing each  $v_j$  must contain span $(v_1, \ldots, v_m)$ . Thus the span of a list of vectors in *V* is the smallest subspace of *V* containing all the vectors in the list.

If span( $\nu_1, ..., \nu_m$ ) equals *V*, we say that ( $\nu_1, ..., \nu_m$ ) *spans V*. A vector space is called *finite dimensional* if some list of vectors in it spans the space. For example,  $\mathbf{F}^n$  is finite dimensional because

 $((1,0,\ldots,0),(0,1,0,\ldots,0),\ldots,(0,\ldots,0,1))$ 

spans  $\mathbf{F}^n$ , as you should verify.

Before giving the next example of a finite-dimensional vector space, we need to define the degree of a polynomial. A polynomial  $p \in \mathcal{P}(\mathbf{F})$  is said to have *degree* m if there exist scalars  $a_0, a_1, \ldots, a_m \in \mathbf{F}$  with  $a_m \neq 0$  such that

**2.2** 
$$p(z) = a_0 + a_1 z + \dots + a_m z^m$$

same as span.

Some mathematicians

use the term linear

Recall that by definition every list has finite length. for all  $z \in \mathbf{F}$ . The polynomial that is identically 0 is said to have degree  $-\infty$ .

For *m* a nonnegative integer, let  $\mathcal{P}_m(\mathbf{F})$  denote the set of all polynomials with coefficients in **F** and degree at most *m*. You should verify that  $\mathcal{P}_m(\mathbf{F})$  is a subspace of  $\mathcal{P}(\mathbf{F})$ ; hence  $\mathcal{P}_m(\mathbf{F})$  is a vector space. This vector space is finite dimensional because it is spanned by the list  $(1, z, \ldots, z^m)$ ; here we are slightly abusing notation by letting  $z^k$  denote a function (so *z* is a dummy variable).

A vector space that is not finite dimensional is called *infinite dimensional*. For example,  $\mathcal{P}(\mathbf{F})$  is infinite dimensional. To prove this, consider any list of elements of  $\mathcal{P}(\mathbf{F})$ . Let *m* denote the highest degree of any of the polynomials in the list under consideration (recall that by definition a list has finite length). Then every polynomial in the span of this list must have degree at most *m*. Thus our list cannot span  $\mathcal{P}(\mathbf{F})$ . Because no list spans  $\mathcal{P}(\mathbf{F})$ , this vector space is infinite dimensional.

The vector space  $\mathbf{F}^{\infty}$ , consisting of all sequences of elements of  $\mathbf{F}$ , is also infinite dimensional, though this is a bit harder to prove. You should be able to give a proof by using some of the tools we will soon develop.

Suppose  $v_1, ..., v_m \in V$  and  $v \in \text{span}(v_1, ..., v_m)$ . By the definition of span, there exist  $a_1, ..., a_m \in \mathbf{F}$  such that

$$\nu = a_1 \nu_1 + \cdots + a_m \nu_m.$$

Consider the question of whether the choice of *a*'s in the equation above is unique. Suppose  $\hat{a}_1, \ldots, \hat{a}_m$  is another set of scalars such that

$$\nu = \hat{a}_1 \nu_1 + \cdots + \hat{a}_m \nu_m.$$

Subtracting the last two equations, we have

$$0 = (a_1 - \hat{a}_1)v_1 + \cdots + (a_m - \hat{a}_m)v_m.$$

Thus we have written 0 as a linear combination of  $(v_1, \ldots, v_m)$ . If the only way to do this is the obvious way (using 0 for all scalars), then each  $a_j - \hat{a}_j$  equals 0, which means that each  $a_j$  equals  $\hat{a}_j$  (and thus the choice of *a*'s was indeed unique). This situation is so important that we give it a special name—linear independence—which we now define.

A list  $(v_1, ..., v_m)$  of vectors in *V* is called *linearly independent* if the only choice of  $a_1, ..., a_m \in \mathbf{F}$  that makes  $a_1v_1 + \cdots + a_mv_m$  equal 0 is  $a_1 = \cdots = a_m = 0$ . For example,

Infinite-dimensional vector spaces, which we will not mention much anymore, are the center of attention in the branch of mathematics called **functional analysis**. Functional analysis uses tools from both analysis and algebra.

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is linearly independent in  $F^4$ , as you should verify. The reasoning in the previous paragraph shows that  $(\nu_1, \ldots, \nu_m)$  is linearly independent if and only if each vector in span $(\nu_1, \ldots, \nu_m)$  has only one representation as a linear combination of  $(\nu_1, \ldots, \nu_m)$ .

For another example of a linearly independent list, fix a nonnegative integer *m*. Then  $(1, z, ..., z^m)$  is linearly independent in  $\mathcal{P}(\mathbf{F})$ . To verify this, suppose that  $a_0, a_1, ..., a_m \in \mathbf{F}$  are such that

$$a_0 + a_1 z + \cdots + a_m z^m = 0$$

for every  $z \in \mathbf{F}$ . If at least one of the coefficients  $a_0, a_1, \ldots, a_m$  were nonzero, then 2.3 could be satisfied by at most m distinct values of z (if you are unfamiliar with this fact, just believe it for now; we will prove it in Chapter 4); this contradiction shows that all the coefficients in 2.3 equal 0. Hence  $(1, z, \ldots, z^m)$  is linearly independent, as claimed.

A list of vectors in *V* is called *linearly dependent* if it is not linearly independent. In other words, a list  $(v_1, ..., v_m)$  of vectors in *V* is linearly dependent if there exist  $a_1, ..., a_m \in \mathbf{F}$ , not all 0, such that  $a_1v_1 + \cdots + a_mv_m = 0$ . For example, ((2, 3, 1), (1, -1, 2), (7, 3, 8)) is linearly dependent in  $\mathbf{F}^3$  because

$$2(2,3,1) + 3(1,-1,2) + (-1)(7,3,8) = (0,0,0).$$

As another example, any list of vectors containing the 0 vector is linearly dependent (why?).

You should verify that a list (v) of length 1 is linearly independent if and only if  $v \neq 0$ . You should also verify that a list of length 2 is linearly independent if and only if neither vector is a scalar multiple of the other. Caution: a list of length three or more may be linearly dependent even though no vector in the list is a scalar multiple of any other vector in the list, as shown by the example in the previous paragraph.

If some vectors are removed from a linearly independent list, the remaining list is also linearly independent, as you should verify. To allow this to remain true even if we remove all the vectors, we declare the empty list () to be linearly independent.

The lemma below will often be useful. It states that given a linearly dependent list of vectors, with the first vector not zero, one of the vectors is in the span of the previous ones and furthermore we can throw out that vector without changing the span of the original list.

Most linear algebra texts define linearly independent sets instead of linearly independent lists. With that definition, the set  $\{(0,1), (0,1), (1,0)\}$  is linearly independent in **F**<sup>2</sup> because it equals the *set* {(0,1), (1,0)}*. With* our definition, the list ((0,1),(0,1),(1,0)) is not linearly independent (because 1 times the first vector plus -1 times the second vector plus 0 times the third vector equals 0). By dealing with lists instead of sets, we will avoid some problems associated with the usual approach.

**2.4** Linear Dependence Lemma: If  $(v_1, ..., v_m)$  is linearly dependent in *V* and  $v_1 \neq 0$ , then there exists  $j \in \{2, ..., m\}$  such that the following hold:

- (a)  $\nu_j \in \operatorname{span}(\nu_1, \ldots, \nu_{j-1});$
- (b) if the j<sup>th</sup> term is removed from (ν<sub>1</sub>,...,ν<sub>m</sub>), the span of the remaining list equals span(ν<sub>1</sub>,...,ν<sub>m</sub>).

**PROOF:** Suppose  $(v_1, ..., v_m)$  is linearly dependent in *V* and  $v_1 \neq 0$ . Then there exist  $a_1, ..., a_m \in \mathbf{F}$ , not all 0, such that

$$a_1\nu_1+\cdots+a_m\nu_m=0.$$

Not all of  $a_2, a_3, ..., a_m$  can be 0 (because  $v_1 \neq 0$ ). Let *j* be the largest element of  $\{2, ..., m\}$  such that  $a_j \neq 0$ . Then

2.5 
$$v_j = -\frac{a_1}{a_j}v_1 - \cdots - \frac{a_{j-1}}{a_j}v_{j-1},$$

proving (a).

To prove (b), suppose that  $u \in \text{span}(v_1, \dots, v_m)$ . Then there exist  $c_1, \dots, c_m \in \mathbf{F}$  such that

$$u=c_1v_1+\cdots+c_mv_m.$$

In the equation above, we can replace  $v_j$  with the right side of 2.5, which shows that u is in the span of the list obtained by removing the  $j^{\text{th}}$  term from  $(v_1, \ldots, v_m)$ . Thus (b) holds.

Now we come to a key result. It says that linearly independent lists are never longer than spanning lists.

**2.6 Theorem:** In a finite-dimensional vector space, the length of every linearly independent list of vectors is less than or equal to the length of every spanning list of vectors.

**PROOF:** Suppose that  $(u_1, \ldots, u_m)$  is linearly independent in *V* and that  $(w_1, \ldots, w_n)$  spans *V*. We need to prove that  $m \le n$ . We do so through the multistep process described below; note that in each step we add one of the *u*'s and remove one of the *w*'s.

Suppose that for each positive integer m, there exists a linearly independent list of m vectors in V. Then this theorem implies that V is infinite dimensional.

## Step 1

The list  $(w_1, \ldots, w_n)$  spans *V*, and thus adjoining any vector to it produces a linearly dependent list. In particular, the list

 $(u_1, w_1, \ldots, w_n)$ 

is linearly dependent. Thus by the linear dependence lemma (2.4), we can remove one of the *w*'s so that the list *B* (of length *n*) consisting of  $u_1$  and the remaining *w*'s spans *V*.

## Step j

The list *B* (of length *n*) from step j-1 spans *V*, and thus adjoining any vector to it produces a linearly dependent list. In particular, the list of length (n + 1) obtained by adjoining  $u_j$  to *B*, placing it just after  $u_1, \ldots, u_{j-1}$ , is linearly dependent. By the linear dependence lemma (2.4), one of the vectors in this list is in the span of the previous ones, and because  $(u_1, \ldots, u_j)$  is linearly independent, this vector must be one of the *w*'s, not one of the *u*'s. We can remove that *w* from *B* so that the new list *B* (of length *n*) consisting of  $u_1, \ldots, u_j$  and the remaining *w*'s spans *V*.

After step *m*, we have added all the *u*'s and the process stops. If at any step we added a *u* and had no more *w*'s to remove, then we would have a contradiction. Thus there must be at least as many *w*'s as *u*'s.

Our intuition tells us that any vector space contained in a finitedimensional vector space should also be finite dimensional. We now prove that this intuition is correct.

**2.7 Proposition:** *Every subspace of a finite-dimensional vector space is finite dimensional.* 

PROOF: Suppose V is finite dimensional and U is a subspace of V. We need to prove that U is finite dimensional. We do this through the following multistep construction.

#### Step 1

If  $U = \{0\}$ , then *U* is finite dimensional and we are done. If  $U \neq \{0\}$ , then choose a nonzero vector  $v_1 \in U$ .

## Step j

If  $U = \text{span}(v_1, \dots, v_{j-1})$ , then *U* is finite dimensional and we are

done. If  $U \neq \text{span}(\nu_1, \dots, \nu_{j-1})$ , then choose a vector  $\nu_j \in U$  such that

$$v_j \notin \operatorname{span}(v_1,\ldots,v_{j-1}).$$

After each step, as long as the process continues, we have constructed a list of vectors such that no vector in this list is in the span of the previous vectors. Thus after each step we have constructed a linearly independent list, by the linear dependence lemma (2.4). This linearly independent list cannot be longer than any spanning list of V (by 2.6), and thus the process must eventually terminate, which means that U is finite dimensional.

## Bases

A *basis* of *V* is a list of vectors in *V* that is linearly independent and spans *V*. For example,

$$((1, 0, \dots, 0), (0, 1, 0, \dots, 0), \dots, (0, \dots, 0, 1))$$

is a basis of  $\mathbf{F}^n$ , called the *standard basis* of  $\mathbf{F}^n$ . In addition to the standard basis,  $\mathbf{F}^n$  has many other bases. For example, ((1,2), (3,5)) is a basis of  $\mathbf{F}^2$ . The list ((1,2)) is linearly independent but is not a basis of  $\mathbf{F}^2$  because it does not span  $\mathbf{F}^2$ . The list ((1,2), (3,5), (4,7)) spans  $\mathbf{F}^2$  but is not a basis because it is not linearly independent. As another example,  $(1, z, \dots, z^m)$  is a basis of  $\mathcal{P}_m(\mathbf{F})$ .

The next proposition helps explain why bases are useful.

**2.8 Proposition:** A list  $(v_1, ..., v_n)$  of vectors in *V* is a basis of *V* if and only if every  $v \in V$  can be written uniquely in the form

$$\nu = a_1 \nu_1 + \cdots + a_n \nu_n,$$

where  $a_1, \ldots, a_n \in \mathbf{F}$ .

**PROOF:** First suppose that  $(v_1, \ldots, v_n)$  is a basis of *V*. Let  $v \in V$ . Because  $(v_1, \ldots, v_n)$  spans *V*, there exist  $a_1, \ldots, a_n \in \mathbf{F}$  such that 2.9 holds. To show that the representation in 2.9 is unique, suppose that  $b_1, \ldots, b_n$  are scalars so that we also have This proof is essentially a repetition of the ideas that led us to the definition of linear independence.

$$\nu = b_1 \nu_1 + \cdots + b_n \nu_n.$$

Subtracting the last equation from 2.9, we get

$$0 = (a_1 - b_1)v_1 + \dots + (a_n - b_n)v_n.$$

This implies that each  $a_j - b_j = 0$  (because  $(v_1, ..., v_n)$  is linearly independent) and hence  $a_1 = b_1, ..., a_n = b_n$ . We have the desired uniqueness, completing the proof in one direction.

For the other direction, suppose that every  $\nu \in V$  can be written uniquely in the form given by 2.9. Clearly this implies that  $(\nu_1, \ldots, \nu_n)$ spans *V*. To show that  $(\nu_1, \ldots, \nu_n)$  is linearly independent, suppose that  $a_1, \ldots, a_n \in \mathbf{F}$  are such that

$$0 = a_1 \nu_1 + \cdots + a_n \nu_n.$$

The uniqueness of the representation 2.9 (with  $\nu = 0$ ) implies that  $a_1 = \cdots = a_n = 0$ . Thus  $(\nu_1, \dots, \nu_n)$  is linearly independent and hence is a basis of *V*.

A spanning list in a vector space may not be a basis because it is not linearly independent. Our next result says that given any spanning list, some of the vectors in it can be discarded so that the remaining list is linearly independent and still spans the vector space.

**2.10** Theorem: *Every spanning list in a vector space can be reduced to a basis of the vector space.* 

**PROOF:** Suppose  $(v_1, ..., v_n)$  spans *V*. We want to remove some of the vectors from  $(v_1, ..., v_n)$  so that the remaining vectors form a basis of *V*. We do this through the multistep process described below. Start with  $B = (v_1, ..., v_n)$ .

## Step 1

If  $v_1 = 0$ , delete  $v_1$  from *B*. If  $v_1 \neq 0$ , leave *B* unchanged.

#### Step j

If  $v_j$  is in span $(v_1, \dots, v_{j-1})$ , delete  $v_j$  from *B*. If  $v_j$  is not in span $(v_1, \dots, v_{j-1})$ , leave *B* unchanged.

Stop the process after step n, getting a list B. This list B spans V because our original list spanned B and we have discarded only vectors that were already in the span of the previous vectors. The process

insures that no vector in *B* is in the span of the previous ones. Thus *B* is linearly independent, by the linear dependence lemma (2.4). Hence *B* is a basis of *V*.

Consider the list

which spans  $F^2$ . To make sure that you understand the last proof, you should verify that the process in the proof produces ((1, 2), (4, 7)), a basis of  $F^2$ , when applied to the list above.

Our next result, an easy corollary of the last theorem, tells us that every finite-dimensional vector space has a basis.

## **2.11 Corollary:** *Every finite-dimensional vector space has a basis.*

PROOF: By definition, a finite-dimensional vector space has a spanning list. The previous theorem tells us that any spanning list can be reduced to a basis.

We have crafted our definitions so that the finite-dimensional vector space  $\{0\}$  is not a counterexample to the corollary above. In particular, the empty list () is a basis of the vector space  $\{0\}$  because this list has been defined to be linearly independent and to have span  $\{0\}$ .

Our next theorem is in some sense a dual of 2.10, which said that every spanning list can be reduced to a basis. Now we show that given any linearly independent list, we can adjoin some additional vectors so that the extended list is still linearly independent but also spans the space.

**2.12 Theorem:** *Every linearly independent list of vectors in a finite- dimensional vector space can be extended to a basis of the vector space.* 

PROOF: Suppose *V* is finite dimensional and  $(v_1, \ldots, v_m)$  is linearly independent in *V*. We want to extend  $(v_1, \ldots, v_m)$  to a basis of *V*. We do this through the multistep process described below. First we let  $(w_1, \ldots, w_n)$  be any list of vectors in *V* that spans *V*.

## Step 1

If  $w_1$  is in the span of  $(v_1, \ldots, v_m)$ , let  $B = (v_1, \ldots, v_m)$ . If  $w_1$  is not in the span of  $(v_1, \ldots, v_m)$ , let  $B = (v_1, \ldots, v_m, w_1)$ .

This theorem can be used to give another proof of the previous corollary. Specifically, suppose V is finite dimensional. This theorem implies that the empty list () can be extended to a basis of V. In particular, V has a basis. Step j

If  $w_j$  is in the span of *B*, leave *B* unchanged. If  $w_j$  is not in the span of *B*, extend *B* by adjoining  $w_j$  to it.

After each step, *B* is still linearly independent because otherwise the linear dependence lemma (2.4) would give a contradiction (recall that  $(v_1, \ldots, v_m)$  is linearly independent and any  $w_j$  that is adjoined to *B* is not in the span of the previous vectors in *B*). After step *n*, the span of *B* includes all the *w*'s. Thus the *B* obtained after step *n* spans *V* and hence is a basis of *V*.

As a nice application of the theorem above, we now show that every subspace of a finite-dimensional vector space can be paired with another subspace to form a direct sum of the whole space.

**2.13 Proposition:** Suppose *V* is finite dimensional and *U* is a subspace of *V*. Then there is a subspace *W* of *V* such that  $V = U \oplus W$ .

**PROOF:** Because *V* is finite dimensional, so is *U* (see 2.7). Thus there is a basis  $(u_1, \ldots, u_m)$  of *U* (see 2.11). Of course  $(u_1, \ldots, u_m)$  is a linearly independent list of vectors in *V*, and thus it can be extended to a basis  $(u_1, \ldots, u_m, w_1, \ldots, w_n)$  of *V* (see 2.12). Let  $W = \text{span}(w_1, \ldots, w_n)$ .

To prove that  $V = U \oplus W$ , we need to show that

$$V = U + W$$
 and  $U \cap W = \{0\};$ 

see 1.9. To prove the first equation, suppose that  $v \in V$ . Then, because the list  $(u_1, \ldots, u_m, w_1, \ldots, w_n)$  spans *V*, there exist scalars  $a_1, \ldots, a_m, b_1, \ldots, b_n \in \mathbf{F}$  such that

$$\nu = \underbrace{a_1u_1 + \cdots + a_mu_m}_{u} + \underbrace{b_1w_1 + \cdots + b_nw_n}_{w}.$$

In other words, we have v = u + w, where  $u \in U$  and  $w \in W$  are defined as above. Thus  $v \in U + W$ , completing the proof that V = U + W.

To show that  $U \cap W = \{0\}$ , suppose  $v \in U \cap W$ . Then there exist scalars  $a_1, \ldots, a_m, b_1, \ldots, b_n \in \mathbf{F}$  such that

$$\nu = a_1u_1 + \cdots + a_mu_m = b_1w_1 + \cdots + b_nw_n.$$

Thus

Using the same basic ideas but considerably more advanced tools, this proposition can be proved without the hypothesis that V is finite dimensional.

$$a_1u_1+\cdots+a_mu_m-b_1w_1-\cdots-b_nw_n=0.$$

Because  $(u_1, \ldots, u_m, w_1, \ldots, w_n)$  is linearly independent, this implies that  $a_1 = \cdots = a_m = b_1 = \cdots = b_n = 0$ . Thus  $\nu = 0$ , completing the proof that  $U \cap W = \{0\}$ .

## Dímension

Though we have been discussing finite-dimensional vector spaces, we have not yet defined the dimension of such an object. How should dimension be defined? A reasonable definition should force the dimension of  $\mathbf{F}^{n}$  to equal n. Notice that the basis

$$((1, 0, \dots, 0), (0, 1, 0, \dots, 0), \dots, (0, \dots, 0, 1))$$

has length n. Thus we are tempted to define the dimension as the length of a basis. However, a finite-dimensional vector space in general has many different bases, and our attempted definition makes sense only if all bases in a given vector space have the same length. Fortunately that turns out to be the case, as we now show.

**2.14 Theorem:** Any two bases of a finite-dimensional vector space have the same length.

**PROOF:** Suppose *V* is finite dimensional. Let  $B_1$  and  $B_2$  be any two bases of *V*. Then  $B_1$  is linearly independent in *V* and  $B_2$  spans *V*, so the length of  $B_1$  is at most the length of  $B_2$  (by 2.6). Interchanging the roles of  $B_1$  and  $B_2$ , we also see that the length of  $B_2$  is at most the length of  $B_1$ . Thus the length of  $B_1$  must equal the length of  $B_2$ , as desired.

Now that we know that any two bases of a finite-dimensional vector space have the same length, we can formally define the dimension of such spaces. The *dimension* of a finite-dimensional vector space is defined to be the length of any basis of the vector space. The dimension of *V* (if *V* is finite dimensional) is denoted by dim *V*. As examples, note that dim  $\mathbf{F}^n = n$  and dim  $\mathcal{P}_m(\mathbf{F}) = m + 1$ .

Every subspace of a finite-dimensional vector space is finite dimensional (by 2.7) and so has a dimension. The next result gives the expected inequality about the dimension of a subspace. **2.15 Proposition:** *If V is finite dimensional and U is a subspace of V, then* dim  $U \le \dim V$ *.* 

**PROOF:** Suppose that *V* is finite dimensional and *U* is a subspace of *V*. Any basis of *U* is a linearly independent list of vectors in *V* and thus can be extended to a basis of *V* (by 2.12). Hence the length of a basis of *U* is less than or equal to the length of a basis of *V*.

To check that a list of vectors in V is a basis of V, we must, according to the definition, show that the list in question satisfies two properties: it must be linearly independent and it must span V. The next two results show that if the list in question has the right length, then we need only check that it satisfies one of the required two properties. We begin by proving that every spanning list with the right length is a basis.

**2.16 Proposition:** *If V is finite dimensional, then every spanning list of vectors in V with length* dim *V is a basis of V*.

**PROOF:** Suppose dim V = n and  $(v_1, ..., v_n)$  spans V. The list  $(v_1, ..., v_n)$  can be reduced to a basis of V (by 2.10). However, every basis of V has length n, so in this case the reduction must be the trivial one, meaning that no elements are deleted from  $(v_1, ..., v_n)$ . In other words,  $(v_1, ..., v_n)$  is a basis of V, as desired.

Now we prove that linear independence alone is enough to ensure that a list with the right length is a basis.

**2.17 Proposition:** If *V* is finite dimensional, then every linearly independent list of vectors in *V* with length dim *V* is a basis of *V*.

**PROOF:** Suppose dim V = n and  $(v_1, \ldots, v_n)$  is linearly independent in V. The list  $(v_1, \ldots, v_n)$  can be extended to a basis of V (by 2.12). However, every basis of V has length n, so in this case the extension must be the trivial one, meaning that no elements are adjoined to  $(v_1, \ldots, v_n)$ . In other words,  $(v_1, \ldots, v_n)$  is a basis of V, as desired.

As an example of how the last proposition can be applied, consider the list ((5,7), (4,3)). This list of two vectors in  $\mathbf{F}^2$  is obviously linearly independent (because neither vector is a scalar multiple of the other).

The real vector space  $\mathbf{R}^2$  has dimension 2; the complex vector space C has dimension 1. As sets, **R**<sup>2</sup> can be identified with C (and addition is the same on both spaces, as is scalar multiplication by real numbers). Thus when we talk about the dimension of a vector space, the role played by the choice of **F** cannot be neglected. Because  $F^2$  has dimension 2, the last proposition implies that this linearly independent list of length 2 is a basis of  $F^2$  (we do not need to bother checking that it spans  $F^2$ ).

The next theorem gives a formula for the dimension of the sum of two subspaces of a finite-dimensional vector space.

**2.18 Theorem:** If  $U_1$  and  $U_2$  are subspaces of a finite-dimensional vector space, then

$$\dim(U_1 + U_2) = \dim U_1 + \dim U_2 - \dim(U_1 \cap U_2).$$

PROOF: Let  $(u_1, ..., u_m)$  be a basis of  $U_1 \cap U_2$ ; thus dim $(U_1 \cap U_2) = m$ . Because  $(u_1, ..., u_m)$  is a basis of  $U_1 \cap U_2$ , it is linearly independent in  $U_1$  and hence can be extended to a basis  $(u_1, ..., u_m, v_1, ..., v_j)$  of  $U_1$  (by 2.12). Thus dim  $U_1 = m + j$ . Also extend  $(u_1, ..., u_m)$  to a basis  $(u_1, ..., u_m, w_1, ..., w_k)$  of  $U_2$ ; thus dim  $U_2 = m + k$ .

We will show that  $(u_1, ..., u_m, v_1, ..., v_j, w_1, ..., w_k)$  is a basis of  $U_1 + U_2$ . This will complete the proof because then we will have

$$dim(U_1 + U_2) = m + j + k$$
  
=  $(m + j) + (m + k) - m$   
=  $dim U_1 + dim U_2 - dim(U_1 \cap U_2).$ 

Clearly span( $u_1, \ldots, u_m, v_1, \ldots, v_j, w_1, \ldots, w_k$ ) contains  $U_1$  and  $U_2$  and hence contains  $U_1 + U_2$ . So to show that this list is a basis of  $U_1 + U_2$  we need only show that it is linearly independent. To prove this, suppose

$$a_1u_1 + \cdots + a_mu_m + b_1v_1 + \cdots + b_iv_i + c_1w_1 + \cdots + c_kw_k = 0,$$

where all the *a*'s, *b*'s, and *c*'s are scalars. We need to prove that all the *a*'s, *b*'s, and *c*'s equal 0. The equation above can be rewritten as

$$c_1w_1 + \cdots + c_kw_k = -a_1u_1 - \cdots - a_mu_m - b_1v_1 - \cdots - b_jv_j,$$

which shows that  $c_1w_1 + \cdots + c_kw_k \in U_1$ . All the *w*'s are in  $U_2$ , so this implies that  $c_1w_1 + \cdots + c_kw_k \in U_1 \cap U_2$ . Because  $(u_1, \ldots, u_m)$  is a basis of  $U_1 \cap U_2$ , we can write

$$c_1w_1+\cdots+c_kw_k=d_1u_1+\cdots+d_mu_m$$

This formula for the dimension of the sum of two subspaces is analogous to a familiar counting formula: the number of elements in the union of two finite sets equals the number of elements in the first set, plus the number of elements in the second set, minus the number of elements in the intersection of the two sets. for some choice of scalars  $d_1, \ldots, d_m$ . But  $(u_1, \ldots, u_m, w_1, \ldots, w_k)$  is linearly independent, so the last equation implies that all the *c*'s (and *d*'s) equal 0. Thus our original equation involving the *a*'s, *b*'s, and *c*'s becomes

$$a_1u_1+\cdots+a_mu_m+b_1v_1+\cdots+b_iv_i=0.$$

This equation implies that all the *a*'s and *b*'s are 0 because the list  $(u_1, \ldots, u_m, v_1, \ldots, v_j)$  is linearly independent. We now know that all the *a*'s, *b*'s, and *c*'s equal 0, as desired.

The next proposition shows that dimension meshes well with direct sums. This result will be useful in later chapters.

**2.19 Proposition:** Suppose *V* is finite dimensional and  $U_1, \ldots, U_m$  are subspaces of *V* such that

$$V = U_1 + \cdots + U_m$$

and

$$\dim V = \dim U_1 + \cdots + \dim U_m.$$

Then  $V = U_1 \oplus \cdots \oplus U_m$ .

**PROOF:** Choose a basis for each  $U_j$ . Put these bases together in one list, forming a list that spans *V* (by 2.20) and has length dim *V* (by 2.21). Thus this list is a basis of *V* (by 2.16), and in particular it is linearly independent.

Now suppose that  $u_1 \in U_1, \ldots, u_m \in U_m$  are such that

$$0=u_1+\cdots+u_m.$$

We can write each  $u_j$  as a linear combination of the basis vectors (chosen above) of  $U_j$ . Substituting these linear combinations into the expression above, we have written 0 as a linear combination of the basis of *V* constructed above. Thus all the scalars used in this linear combination must be 0. Thus each  $u_j = 0$ , which proves that  $V = U_1 \oplus \cdots \oplus U_m$ (by 1.8).

Recall that direct sum is analogous to disjoint union. Thus 2.19 is analogous to the statement that if a finite set B is written as  $A_1 \cup \cdots \cup A_m$  and the sum of the number of elements in the A's equals the number of elements in B, then the union is a disjoint union.

## Exercíses

1. Prove that if  $(v_1, \ldots, v_n)$  spans *V*, then so does the list

$$(v_1 - v_2, v_2 - v_3, \dots, v_{n-1} - v_n, v_n)$$

obtained by subtracting from each vector (except the last one) the following vector.

2. Prove that if  $(v_1, ..., v_n)$  is linearly independent in *V*, then so is the list

$$(v_1 - v_2, v_2 - v_3, \dots, v_{n-1} - v_n, v_n)$$

obtained by subtracting from each vector (except the last one) the following vector.

- 3. Suppose  $(v_1, ..., v_n)$  is linearly independent in *V* and  $w \in V$ . Prove that if  $(v_1 + w, ..., v_n + w)$  is linearly dependent, then  $w \in \text{span}(v_1, ..., v_n)$ .
- 4. Suppose *m* is a positive integer. Is the set consisting of 0 and all polynomials with coefficients in **F** and with degree equal to *m* a subspace of  $\mathcal{P}(\mathbf{F})$ ?
- 5. Prove that  $\mathbf{F}^{\infty}$  is infinite dimensional.
- 6. Prove that the real vector space consisting of all continuous realvalued functions on the interval [0, 1] is infinite dimensional.
- 7. Prove that *V* is infinite dimensional if and only if there is a sequence  $v_1, v_2, \ldots$  of vectors in *V* such that  $(v_1, \ldots, v_n)$  is linearly independent for every positive integer *n*.
- 8. Let *U* be the subspace of  $\mathbf{R}^5$  defined by

$$U = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbf{R}^5 : x_1 = 3x_2 \text{ and } x_3 = 7x_4\}$$

Find a basis of U.

- 9. Prove or disprove: there exists a basis  $(p_0, p_1, p_2, p_3)$  of  $\mathcal{P}_3(\mathbf{F})$  such that none of the polynomials  $p_0, p_1, p_2, p_3$  has degree 2.
- 10. Suppose that *V* is finite dimensional, with dim V = n. Prove that there exist one-dimensional subspaces  $U_1, \ldots, U_n$  of *V* such that

$$V=U_1\oplus\cdots\oplus U_n.$$

- 11. Suppose that *V* is finite dimensional and *U* is a subspace of *V* such that dim  $U = \dim V$ . Prove that U = V.
- 12. Suppose that  $p_0, p_1, ..., p_m$  are polynomials in  $\mathcal{P}_m(\mathbf{F})$  such that  $p_j(2) = 0$  for each *j*. Prove that  $(p_0, p_1, ..., p_m)$  is not linearly independent in  $\mathcal{P}_m(\mathbf{F})$ .
- 13. Suppose *U* and *W* are subspaces of  $\mathbb{R}^8$  such that dim U = 3, dim W = 5, and  $U + W = \mathbb{R}^8$ . Prove that  $U \cap W = \{0\}$ .
- 14. Suppose that *U* and *W* are both five-dimensional subspaces of  $\mathbb{R}^9$ . Prove that  $U \cap W \neq \{0\}$ .
- 15. You might guess, by analogy with the formula for the number of elements in the union of three subsets of a finite set, that if  $U_1, U_2, U_3$  are subspaces of a finite-dimensional vector space, then

$$dim(U_1 + U_2 + U_3)$$
  
= dim U<sub>1</sub> + dim U<sub>2</sub> + dim U<sub>3</sub>  
- dim(U<sub>1</sub> \cap U\_2) - dim(U<sub>1</sub> \cap U\_3) - dim(U<sub>2</sub> \cap U\_3)  
+ dim(U<sub>1</sub> \cap U<sub>2</sub> \cap U<sub>3</sub>).

Prove this or give a counterexample.

16. Prove that if *V* is finite dimensional and  $U_1, \ldots, U_m$  are subspaces of *V*, then

 $\dim(U_1 + \cdots + U_m) \leq \dim U_1 + \cdots + \dim U_m.$ 

17. Suppose *V* is finite dimensional. Prove that if  $U_1, \ldots, U_m$  are subspaces of *V* such that  $V = U_1 \oplus \cdots \oplus U_m$ , then

 $\dim V = \dim U_1 + \cdots + \dim U_m.$ 

This exercise deepens the analogy between direct sums of subspaces and disjoint unions of subsets. Specifically, compare this exercise to the following obvious statement: if a finite set is written as a disjoint union of subsets, then the number of elements in the set equals the sum of the number of elements in the disjoint subsets.