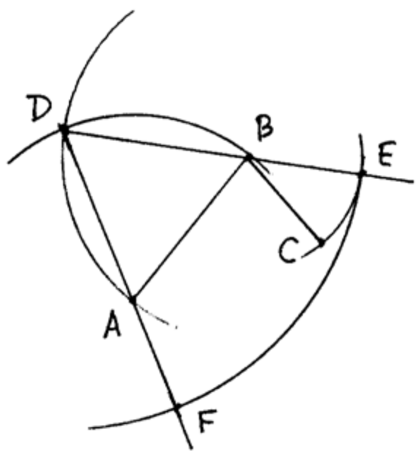
result depends on its position in the logical sequence of propositions. In the case of (I.1), there are no previous propositions, so Euclid's proof depends only on the definitions, postulates, and common notions set out at the beginning of Book I. His proof says, in substance, that $AC = AB$ because they are both radii of the first circle, and $BA = BC$ because they are radii of the second circle, so $AB = AC = BC$ and hence the triangle is equilateral.



Next, let us look at (I.2). Given a point A and a line segment BC, we must construct a line segment AF originating at A, equal to BC. Euclid's method is as follows: Draw AB. Construct the equilateral triangle ABD using the construction of (I.1). Then with center B and radius BC draw a circle to meet DB extended at E. With center D and radius DE draw a circle to meet DA extended at F. Then AF is the required line segment.

The proof is natural enough: $BC = BE$ by construction; $DE = DF$ by construction; $DB = DA$ by construction, so by subtraction $AF = BE = BC$ as required.

But the question that immediately arises is, why did Euclid go to all this trouble when he could have made a much simpler construction: Set the compass points to the distance BC, then draw a circle with center A and radius BC, choose F any point on that circle, and join A to F? We must infer from the presence of this construction that Euclid allowed himself to use the compass only in its narrow sense to draw a circle with a given center and passing through a given point. It could not be lifted off the paper and used to transport a given distance to another location. So some people call Euclid's compass a *collapsible compass*: when you lift it off the paper the points fall together and do not preserve the radius they were set at. However, the function of this construction (I.2) is to show that with the collapsible compass one can still accomplish the same result, *as if the compass had not been collapsible*, namely, to transport a distance to another point in the plane. So from now on, we will allow ourselves to use the compass in this stronger sense, to draw a circle with given center and radius equal to any given line segment.

Counting Steps

To increase our awareness of the process of ruler and compass constructions, let us make precise exactly how the tools can be used, and let us set up a way of counting our steps as a measure of the complexity of the construction. The number of steps needed for a construction is not really important of itself, but by counting our steps we become more conscious of the process. This is one of the practical aspects of this course, to have some fun while we are pondering the deeper theoretical questions.

In any construction problem there are usually some points, lines, or circles given at the outset. The ruler may be used to extend a given or previously constructed line in either direction. The ruler may be used to draw a new line through two distinct points either given or constructed earlier. The ruler may not be used to measure distances, and it may not have any markings on it (hence the frequently used term *straightedge* to emphasize that it may be used only to draw straight lines).

The compass may be used to draw a circle with center a given or previously constructed point, and with radius equal to the distance between any two given or previously constructed points.

In addition, at any time one may choose a point at random, or subject to conditions such as that it should lie on a given line or circle, or be on the other side of a line from a given point, etc.

Each time a new line or circle is drawn, those points in which it intersects previously given or constructed lines and circles will be considered to be constructed also.

For counting, we consider each use of the ruler to construct a new line as one step, and each use of the compass to construct a new circle as one step. Extending lines previously given or constructed, choosing points at random, and obtaining new points as intersections do not count as separate steps.

Thus for example, the construction of the equilateral triangle (I.1) above takes four steps:

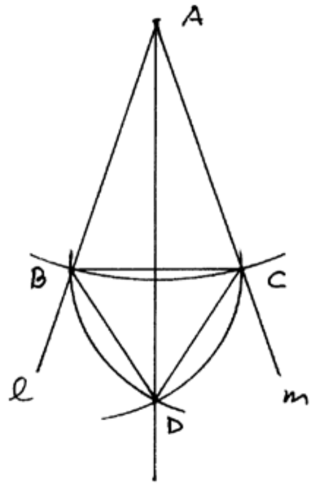
The line segment AB is given
1. Draw circle with center A and radius AB.
2. Draw circle with center B and radius BA. Get C.
3. Draw AC.
4. Draw BC.

Then ABC is the required triangle.

When performing more complicated constructions, we will count all of the steps required to perform the entire construction, so that each construction is self-contained and independent of other constructions (though inevitably each construction will contain elements of other constructions). This imposes a different notion of economy of construction from Euclid's. For while Euclid in his sequential development of the propositions finds it most economical to utilize previous constructions, we will find that minimizing the total number of steps will often lead us to different constructions.

Look at (I.9), for example, to bisect a given angle. The angle is given by a point A and two rays l, m emanating from A. Euclid's method is this: Choose B on l at random. Find C on m such that $AB = AC$ (I.3). Draw BC. Construct the equilateral triangle BCD (I.1). Join AD. Then AD is the angle bisector.

Euclid's method is economical for him because it makes use of previously described constructions (I.3) and (I.1). If we count the number of steps to carry out this construction, we find seven:



Choose B at random on l (no step)
1. Circle center A radius AB, get C.
2. Draw BC.
3. Circle center B radius BC.
4. Circle center C radius CB, get D.
5. Draw BD.
6. Draw CD.
7. Draw AD, which is the angle bisector.

If we are concerned only with making an independent construction for the angle bisector, there is no need to draw the lines BC, BD, CD. Thus the construction reduces to four steps. In order to prove that this construction works, we might want to draw the lines BC, BD, CD and argue as Euclid did. The lines then become part of the proof. But they are not part of the construction, so the construction still requires only four steps.

For another example, look at Euclid's construction (I.10) to bisect a given line segment. He first appeals to (I.1) to construct an equilateral triangle, and then to (I.9) to bisect the angle at its vertex. This is an elegant method, making use of what he has done before. But in terms of numbers of steps, it is not efficient. If we add the numbers of steps used in the two previous results, we get 11 steps. If we make use of points already constructed in (I.1) when we do the construction of (I.9), this reduces to 9. But it is possible to give a direct construction of the midpoint of a segment in only three steps (see Exercise 2.2).

A Note About Accuracy and Exactness of Constructions

When carrying out ruler and compass constructions, we attempt to make our drawings as *accurate* as possible. Using a sharp pencil we draw fine lines and make them pass through given points as closely as possible. Nevertheless, there is always a small error in each step, and those errors will compound throughout a long construction, so that the final figure does not always do just what you want. For example, in constructing the circle circumscribed about a given triangle (Exercise 2.10), you may find that your circle passes nicely through two of the points but misses the third one slightly. This error is inevitable in any drawings we make.

But, to paraphrase the quotation from Plato in Section 1, it is not the line and the circle drawn on the paper that we are thinking of, it is the absolute line and the absolute circle. And in this sense, our construction must be mathematically *exact*. In other words, it must be possible to prove using the reasoning of abstract geometry that this construction in its ideal form gives the exact result we are seeking.

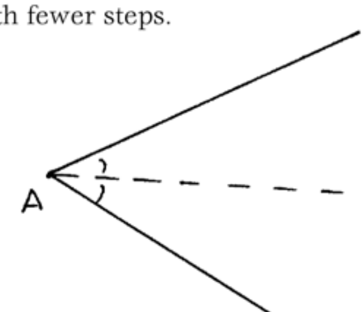
This distinction has caused considerable confusion among amateur mathematicians through the ages, who were trying to make constructions, now known to be impossible, of trisecting the angle or squaring the circle. For many of their constructions are remarkably accurate, while failing to be mathematically exact. (See the interesting book of Dudley (1987), as well as Sections 25, 28 below.)

Exercises

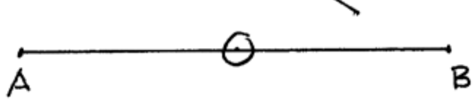
For each of the following problems, carry out a ruler and compass construction as accurately as you can. Number and label each of your steps as in the text. Feel free to use abbreviations such as "*AB*" for "draw a line *AB*"; "⊙*AB*" to draw a circle with center *A* and radius *AB*; or "⊙cAr*BC*" *to draw a circle with center A* and radius *BC*. Label each new point as it is constructed and mention it (e.g., "get *F*") in the appropriate step. For the time being, we are not concerned with the proofs. Just do the construction. You should, however, be able to give an informal proof (convincing argument) of why it works, if asked.

After you make your construction, locate the corresponding proposition in Euclid (Book I, III, or IV) and compare. How many steps does his method require? What do you think is the least number of steps possible? I will sometimes give a *par value* for a construction, which is the typical number of steps an experienced constructor would need. By trying harder, you can sometimes succeed with fewer steps.

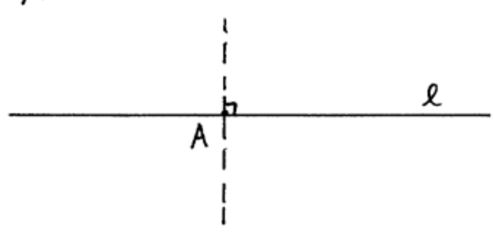
2.1 Given an angle, construct the angle bisector (par = 4).



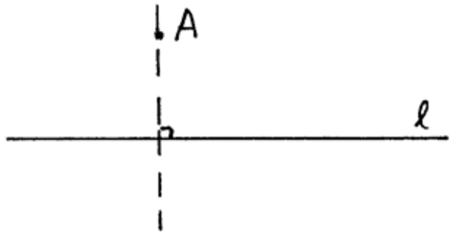
2.2 Given a line segment, find the midpoint of that segment (par = 3).



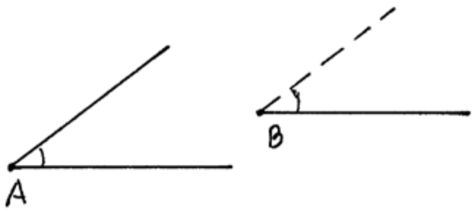
2.3 Given a line l and a point A on l, construct a line perpendicular to l through A (par = 4, possible in 3).



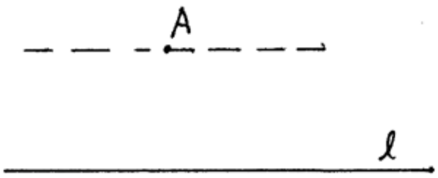
2.4 Given a line l and a point A not on l, construct a line perpendicular to l passing through A (par = 4, possible in 3).



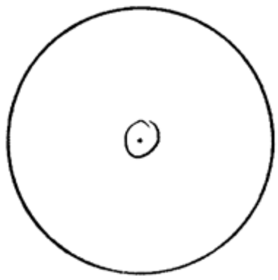
2.5 Given an angle at a point A, and given a ray emanating from a point B, construct an angle at B equal to the angle at A (par = 4).



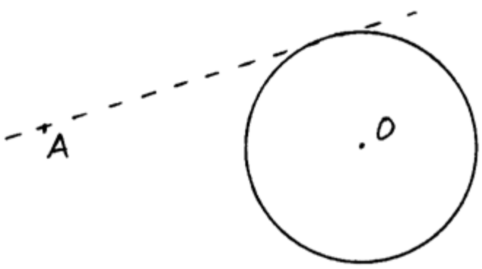
2.6 Given a line l and a point A not on l, construct a line parallel to l, passing through A (par = 3).



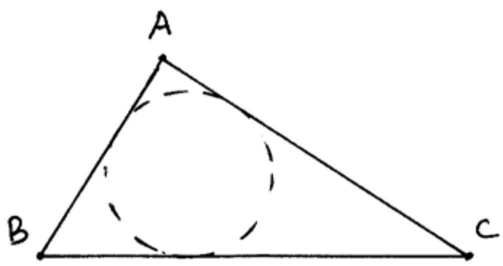
2.7 Given a circumference of a circle, find the center of the circle (par = 5).



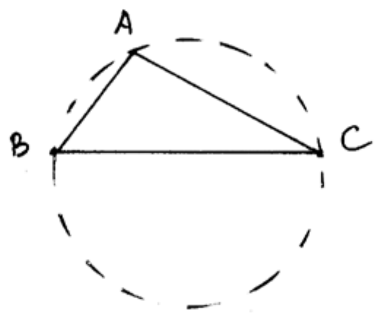
2.8 Given a circle with its center O, and given a point A outside the circle, construct a line through A tangent to the circle. (*Warning*: You may not slide the ruler until it seems to be tangent to the circle. You must construct another point on the desired tangent line before drawing the tangent.) (Par = 6.)



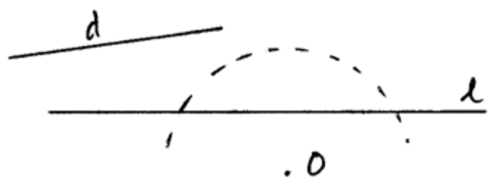
2.9 Construct a circle inscribed in a given triangle ABC (par = 13).



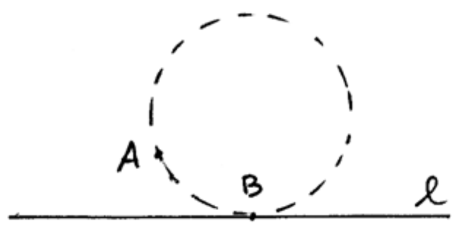
2.10 Construct a circle circumscribed about a given triangle ABC (par $=7$).



2.11 Given a line l, a line segment d, and a point O, construct a circle with center O that cuts off a segment congruent to d on the line l (par $=9$).

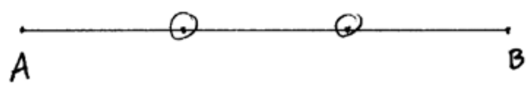


2.12 Given a point A, a line l, and a point B on l, construct a circle that passes through A and is tangent to the line l at B (par $=8$).



2.13 Construct three circles, each one meeting the other two at right angles. (We say that two circles meet at right angles if the radii of the two circles to a point of intersection make right angles.) (Par $=10$.)

2.14 Given a line segment AB, divide it into three equal pieces (par $=6$).



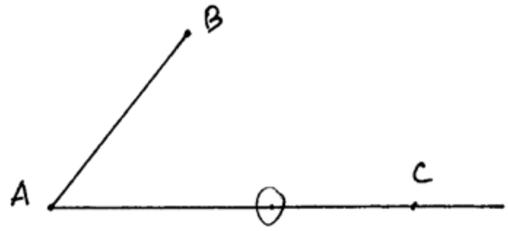
2.15 (The one-inch ruler.) Suzie's ruler broke into little pieces, so she can only draw lines one inch long. Fortunately, her compass is still working. She has two points on her paper approximately 3 inches apart. Help her construct the straight line joining those two points.

2.16 (The rusty compass.) Joe's compass has rusted into a fixed position, so it can only draw circles whose radius is one inch. Fortunately, his ruler is still working. Help him construct an equilateral triangle on a segment AB that is approximately $2\frac{1}{2}$ inches long (par $=6$).

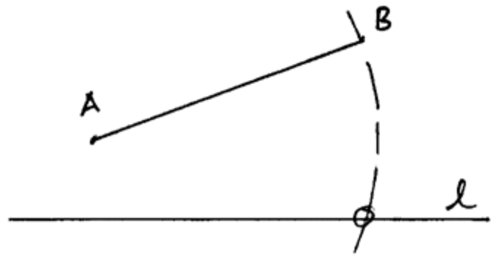
2.17 Using a ruler and rusty compass (cf. Exercise 2.16), construct the perpendicular to a line l at a point A on l (par $=6$).

2.18 Using a ruler and rusty compass, given a line l and a point A more than 2 inches away from l, construct the line through A and perpendicular to l (par $=12$).

2.19 Using a ruler and rusty compass, given a segment AB and given a ray AC, construct a point D on the ray AC such that $AB \cong AD$.

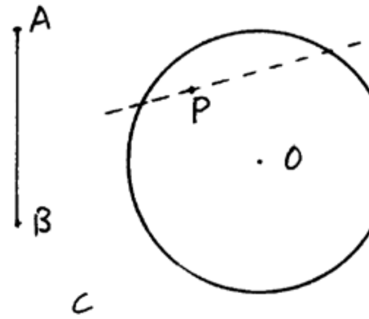


2.20 Using a ruler and rusty compass, given a line l and given a segment AB more than one inch long, construct one of the points C in which the circle of center A and radius AB meets l.

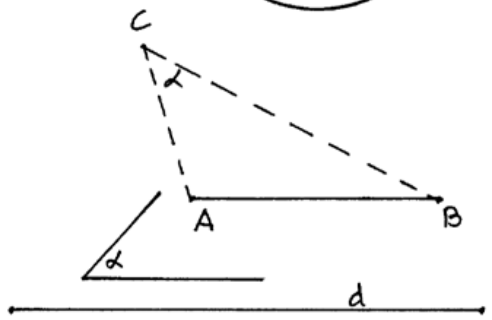


2.21 Discussion question: Is it possible with ruler and rusty compass to construct any figure that can be constructed with ruler and regular compass? What would you need to know in order to *prove* that this is possible? For starters, can you carry out all the constructions of Euclid, Book I, with ruler and rusty compass?

2.22 (Back to regular ruler and compass construction.) Given a segment AB, given a circle with center O, and given a point P inside O, construct (if possible) a line through P on which the circle cuts off a segment congruent to AB (par = 5).



2.23 Given a segment AB, given an angle α, and given another segment d, construct a triangle ABC with base equal to AB, angle α at C, and such that $AC + BC = d$.



2.24 Given two circles Γ, Γ', with centers O, O', construct a line tangent to both circles.