Síntese de Imagens Digitais

SAMPLING AND RECONSTRUCTION

IN COMPUTER GRAPHICS

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Slide 1

The sampling and reconstruction processes in computer graphics correspond to transformations between continuous and discrete representations of a picture.

Errors introduced by these processes can seriously affect the quality of the displayed image.

Some defects in the displayed image:

- jagged appearance of lines and edges of a surface
- visible gaps can be seen in images of containing “slivers”
- when lines or curves are closely spaced - and so with high spatial frequency - gaps can cause Moiré patterns
- scintillation due to small moving objects appearing and disappearing in a dynamic display
- loss of sharpness on lines and edges of a surface.
- ringing: rippling pattern radiating from the edges of a scene

Slide 2

- device dependence: an image computed for one device does not look its best when displayed on another device

Slide 3

$I : \mathbb{R}^2 \longrightarrow \mathcal{C}$	Scene
$\hat{I} : \mathbb{Z}^2 \longrightarrow \mathcal{C}_k, \quad \hat{I}(i,j) = \vec{I}_{ij}$	Discrete Image
$\mathcal{I} : \mathbb{R}^2 \longrightarrow \mathcal{C}, \quad \mathcal{I}(x,y) = \sum \vec{I}_{ij} B_{ij}(x,y)$	Displayed Image

$$\underbrace{I \xrightarrow{Sampling+Quantization} \hat{I}}_{Rendering} \xrightarrow{Reconstruction} \mathcal{I}$$

Rendering should take into consideration both the sampling and the reconstruction processes: $I \overset{?}{\approx} \mathcal{I}$.

- Essentially, sampling is device independent (modulo device resolution)
- Reconstruction process is not device independent.
- Errors due to both processes can introduce visible defects in the displayed image.

Slide 4

Ideal Sampling and Reconstruction

¿ Ideal conditions under which signals such as *colour intensity functions* can be sampled and reconstructed without *loss or distortion of information* ?

Answer: Shannon Theorem [Sha49] (from Whittaker [Whi15])
Continuous Signal: function of continuous variables [Papoulis84]

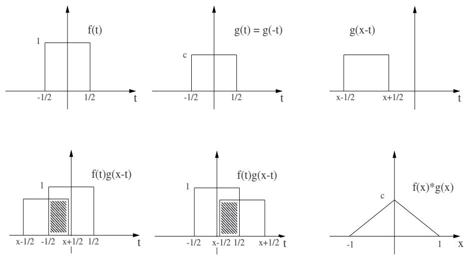
Basic tools for the demonstration of the Shannon theorem:

- Fourier Transform
- Convolution Theorem

Slide 5

Convolution of two signals f and g:

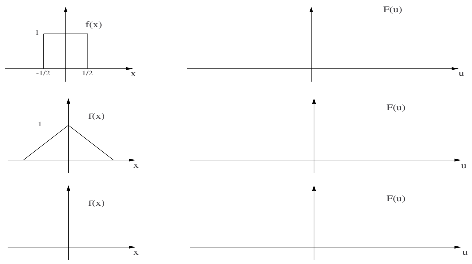
$$(f * g)(x) = \int_{-\infty}^{\infty} f(t)g(x-t)dt \qquad x \in \mathbb{R}$$



Slide 6

Fourier transform (F) of a signal $f : \mathbb{R} \longrightarrow \mathbb{R}$:

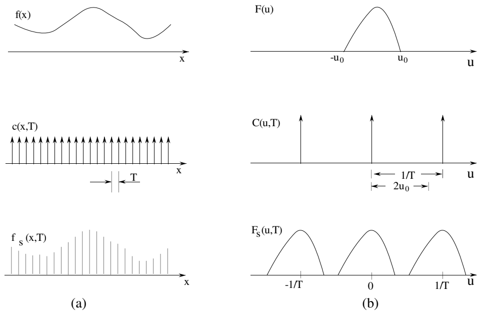
$$F(u) = \int_{-\infty}^{\infty} f(x) e^{-j2\pi ux} dx \qquad u \in \mathbb{R}$$



Slide 7

Convolution Theorem:

$$f_s(x,T) = f(x)\,c\,(x,T) \quad \Longleftrightarrow \quad F_s(u,T) = F(u) * C(u,T)$$



Slide 8

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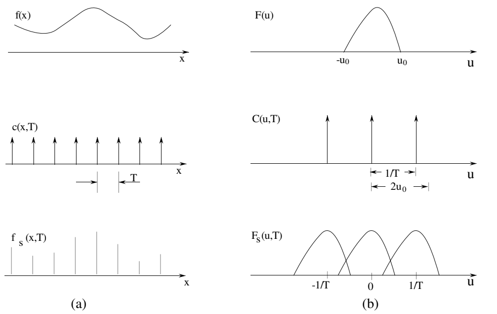


Slide 9

Problem now: *formulate conditions on H and T such that*

$$F(u) = F_r(u,T) \stackrel{def}{=} F_s(u,T)H(u)$$

Solved the problem ...

$$F(u) = F_s(u,T)H(u) \overset{Convolution\,Theorem}{\Longrightarrow} f(x) = f_s(x,T) * h(x)$$

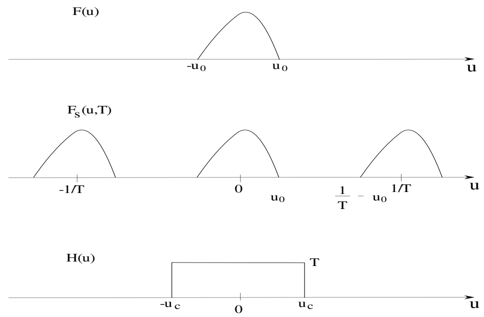
Suppose *there exists* $u_0 \in \mathbb{R}$ *such that* $F(u) = 0$ for $|u| \geq u_0$

and take/define

- T such that $1/T > 2u_0$
- $H(u) = \begin{cases} T\,, & \text{if} \quad |u| \geq u_c \qquad (u_0 < u_c < \frac{1}{T} - u_0) \\ 0\,, & \text{otherwise} \end{cases}$

Slide 10

Ideal low-pass filtering H for exact reconstruction of the original spectrum F from its samples



Slide 11

Working out ...

Inverse Fourier transform of

$$H(u) = \begin{cases} T\,, & \text{if} \quad |u| \geq u_c \\ 0\,, & \text{otherwise} \end{cases}$$

is

$$h(x) \;=\; \int_{-u_c}^{u_c} T\, e^{j2\pi ux} du \;=\; T\, \frac{sin 2\pi u_c x}{\pi x} \;\stackrel{def}{=}\; 2u_c T\, \underbrace{sinc(2u_c x)}$$

Slide 12

... Working out ...

$$f(x) \stackrel{f=f_s*h}{=} \int_{-\infty}^{\infty} f_s(u,T)\, h(x-u)\, du \stackrel{f_s(u)=\sum f(u)\delta(u-mT)}{=}$$

$$\sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} \underbrace{f(u)\, h(x-u)}_{g}\, \delta(u-mT)\, du =$$

$$\stackrel{\int_{-\infty}^{\infty} g(u)\delta(u)du=g(0)}{=} \sum_{m=-\infty}^{\infty} f(mT)\, h(x-mT)$$

Then

$$\sum_{m=-\infty}^{\infty} \underbrace{f(mT)}_{samples}\, \overbrace{h(x-mT)}^{filter} \Longrightarrow f(x)$$

where the reconstruction filter h is given by

$$h(x) = 2u_c T sinc(2u_c x)$$

Slide 13

Summary

Theorem 1 (Shannon) *If the Fourier transform of a signal $f(x)$ is zero for $|u| \geq u_0$ and the signal is periodically sampled with period $T < \frac{1}{2u_0}$ then $f(x)$ can be exactly reconstructed from its samples using the interpolation formula*

$$f(x) \;=\; \sum_{m=-\infty}^{\infty} f(mT)\, h(x-mT) \tag{1}$$

where the reconstruction filter h is given by

$$h(x) = 2u_c T sinc(2u_c x)$$

with u_c satisfying $u_0 < u_c < \frac{1}{T} - u_0$.

Slide 14

Comments and Notation

The upper bound $\frac{1}{2u_0}$ is referred to as the Nyquist period. Its reciprocal is called the Nyquist frequency.

The sampling theorem states that a signal periodically sampled with period less than Nyquist period can be exactly reconstructed by low-pass filtering the sampled signal.

However, if the sampling period is greater than the Nyquist period, the information within the support of H is contaminated by the tails of the higher and lower replications of the original Fourier transform. Then, the original spectrum is no longer recoverable by low-pass filtering.

This phenomenon - caused by undersampling the signal f - is referred to as aliasing.