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The exponentiated generalized inverse Gaussian distribution

Artur J. Lemonte[a,*], Gauss M. Cordeiro[b]

[a] Departamento de Estatística, Universidade de São Paulo, Brazil

[b] Departamento de Estatística, Universidade Federal Rural de Pernambuco, Brazil

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ABSTRACT

The modeling and analysis of lifetime data is an important aspect of statistical work in a wide variety of scientific and technological fields. Good (1953) introduced a probability distribution which is commonly used in the analysis of lifetime data. For the first time, based on this distribution, we propose the so-called exponentiated generalized inverse Gaussian distribution, which extends the exponentiated standard gamma distribution (Nadarajah and Kotz, 2006). Various structural properties of the new distribution are derived, including expansions for its moments, moment generating function, moments of the order statistics, and so forth. We discuss maximum likelihood estimation of the model parameters. The usefulness of the new model is illustrated by means of a real data set.

1. Introduction

The generalized inverse Gaussian (GIG) distribution introduced by Good (1953) is widely used for modeling and analyzing lifetime data. A random variable X has a GIG distribution if its probability density function (pdf) is given by

$$f(x;\lambda,\chi,\psi)=\frac{(\psi/\chi)^{\lambda/2}}{2K_\lambda(\sqrt{\chi\psi})}x^{\lambda-1}\exp\left\{-\frac{1}{2}\left(\psi x+\frac{\chi}{x}\right)\right\},\quad x>0.$$

Here, $-\infty<\lambda<\infty$, $(\chi,\psi)\in\Theta_\lambda$, where $\Theta_\lambda=\{(\chi,\psi):\chi\geq 0,\psi>0\}$ if $\lambda>0$, $\{(\chi,\psi):\chi>0,\psi>0\}$ if $\lambda=0$ and $\{(\chi,\psi):\chi>0,\psi\geq 0\}$ if $\lambda<0$. Also, $K_\nu(z)$ denotes the modified Bessel function of the third kind with index ν and argument z (see, for example, Watson, 1995). Special sub-models include the gamma distribution ($\chi=0,\lambda>0$), the reciprocal gamma distribution ($\psi=0,\lambda<0$), the inverse Gaussian distribution ($\lambda=-1/2$) and the hyperbola distribution ($\lambda=0$). Introducing the parameters $\omega=\chi/2$ and $\eta=\psi/2$, the above density function becomes

$$f(x;\lambda,\omega,\eta)=C\,x^{\lambda-1}\exp\{-(\eta x+\omega x^{-1})\},\quad x>0,\tag{1}$$

where the normalizing constant is $C=C(\lambda,\omega,\eta)=(\eta/\omega)^{\lambda/2}/\{2K_\lambda(2\sqrt{\eta\omega})\}$. A random variable X having density function (1) is denoted by $X\sim\mathrm{GIG}(\lambda,\omega,\eta)$. The rth moment of X about zero is given by

$$\mathbb{E}(X^r)=\left(\frac{\omega}{\eta}\right)^{r/2}\frac{K_{\lambda+r}(2\sqrt{\eta\omega})}{K_\lambda(2\sqrt{\eta\omega})}.\tag{2}$$

* Corresponding author.
E-mail address: arturlemonte@gmail.com (A.J. Lemonte).

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The moment generating function (mgf) associated to (1) is

$$M(t) = \left(\frac{\eta}{\eta - t}\right)^{\lambda/2} \frac{K_\lambda\left(2\sqrt{(\eta - t)\omega}\right)}{K_\lambda\left(2\sqrt{\omega\eta}\right)}. \quad (3)$$

In addition, the cumulative distribution function (cdf) becomes

$$F(x) = F(x; \lambda, \omega, \eta) = C\eta^{-\lambda}\gamma(\lambda, \eta x; \eta\omega) = 1 - C\eta^{-\lambda}\Gamma(\lambda, \eta x; \eta\omega), \quad (4)$$

where $\gamma(\alpha, x; b) = \int_0^x t^{\alpha-1}\exp\{-(t + bt^{-1})\}dt$ and $\Gamma(\alpha, x; b) = \int_x^\infty t^{\alpha-1}\exp\{-(t + bt^{-1})\}dt$ are the generalized incomplete gamma functions (Chaudhry and Zubair, 1994) discussed in detail by Chaudhry and Zubair (2002). If the argument b vanishes, the functions $\gamma(\alpha, x; b)$ and $\Gamma(\alpha, x; b)$ reduce to the ordinary incomplete gamma functions $\gamma(\alpha, x; 0) = \gamma(\alpha, x) = \int_0^x t^{\alpha-1}e^{-t}dt$ and $\Gamma(\alpha, x; 0) = \Gamma(\alpha, x) = \int_x^\infty t^{\alpha-1}e^{-t}dt$. The GIG distribution has survival and hazard rate functions given by $S(x) = C\eta^{-\lambda}\Gamma(\lambda, \eta x; \eta\omega)$ and $h(x) = \eta^\lambda x^{\lambda-1}\exp\{-(\eta x+\omega x^{-1})\}/\Gamma(\lambda, \eta x; \eta\omega)$, respectively.

From Chaudhry and Zubair (2002, Eq. (2.91)), we have $\Gamma(\alpha, x; b) = \sum_{j=0}^\infty \Gamma(\alpha - j, x)(-b)^j/j!$. Since $\gamma(\alpha, x) = \sum_{k=0}^\infty (-1)^k x^{k+\alpha}/\{k!(k + \alpha)\}$ and $\Gamma(\alpha, x) = \Gamma(\alpha) - \gamma(\alpha, x)$, where $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1}e^{-t}dt$ is the gamma function, the function $\Gamma(\alpha, x; b)$ can be expanded as $\Gamma(\alpha, x; b) = \sum_{j=0}^\infty (-b)^j[\Gamma(\alpha - j)/j! - \sum_{k=0}^\infty (-1)^k x^{k+\alpha-j}/\{k!j!(k + \alpha - j)\}]$. Thus, inserting this equation in (4), $F(x)$ can be rewritten as

$$F(x) = 1 - \left(\rho + \sum_{j,k=0}^\infty a_{j,k}x^{k-j+\lambda}\right), \quad (5)$$

where $\rho = \rho(\lambda, \eta, \omega) = C\eta^{-\lambda}\sum_{j=0}^\infty \Gamma(\lambda - j)(-\eta\omega)^j/j!$ and $a_{j,k} = a_{j,k}(\lambda, \eta, \omega) = (-1)^{k+j+1}C\eta^k\omega^j/\{(k - j + \lambda)j!k!\}$. To calculate ρ, the index j can stop after a large number of summands.

The GIG distribution has been applied in a variety of fields of statistics; see, for example, Embrechts (1983), Iyengar and Liao (1997) and Thabane and Haq (1999). Sichel (1975) used the GIG distribution to construct mixture of Poisson distributions. Barndorff-Nielsen (1978) and Barndorff-Nielsen et al. (1978) used the GIG distribution as a mixing distribution to obtain the generalized hyperbolic distribution as a mixture of normal distributions. Statistical properties and distributional behavior of the GIG distribution are discussed by Jørgensen (1982). Atkinson (1982) and Dagpunar (1989) provided algorithms for simulating the GIG distribution. More recently, Nguyen et al. (2003) showed that the GIG distribution has positive skewness.

In this note, we introduce the so-called exponentiated generalized inverse Gaussian (EGIG) distribution that contains the GIG model and other special models. Additionally, we introduce the exponentiated gamma distribution, which generalizes the exponentiated standard gamma distribution proposed by Nadarajah and Kotz (2006). We obtain some mathematical properties and discuss maximum likelihood estimation of the parameters. The rest of the article is organized as follows. In Section 2, we introduce the new distribution. Expansions for the quantile functions of the GIG and EGIG distributions are presented in Section 3. Section 4 gives a formal expansion for the EGIG density function as a mixture of GIG density functions. The moments, moment generating function (mgf), moments of the order statistics and L-moments are also derived in this section. Mean deviations and Rényi entropy are investigated in Section 5. The reliability is determined in Section 6. We discuss in Section 7 maximum likelihood estimation of the model parameters. Section 8 gives an application to a real data set to show that the proposed distribution can be used quite effectively in analyzing lifetime data. Section 9 provides some conclusions.

2. The exponentiated generalized inverse Gaussian distribution

Since 1995, the exponentiated distributions have been widely studied in statistics and numerous authors have developed various classes of these distributions. Mudholkar et al. (1995) proposed the exponentiated Weibull distribution. Its properties have been studied in more detail by Mudholkar and Hutson (1996) and Nassar and Eissa (2003). Gupta and Kundu (1999) introduced the exponentiated exponential distribution as a generalization of the standard exponential distribution. Nadarajah and Kotz (2006) proposed, based on the same idea, four more exponentiated type distributions to extend the standard gamma, standard Weibull, standard Gumbel and standard Fréchet distributions. More recently, Gusmão et al. (2009) introduced the exponentiated inverse Weibull distribution. In the same way, we generalize the GIG distribution.

Let $F(x)$ be the cdf of the GIG distribution (Good, 1953). The EGIG cdf can be defined by elevating $F(x)$ to the power $\beta > 0$, i.e. $G(x) = F(x)^\beta$. Hence, the cdf and density function of the EGIG distribution with four parameters, say EGIG$(\lambda, \omega, \eta, \beta)$, are given, respectively, by

$$G(x) = \left\{1 - C\eta^{-\lambda}\Gamma(\lambda, \eta x; \eta\omega)\right\}^\beta, \quad x > 0, \quad (6)$$

and

$$g(x) = \beta Cx^{\lambda-1}\exp\left\{-\left(\eta x + \omega x^{-1}\right)\right\}\left\{1 - C\eta^{-\lambda}\Gamma(\lambda, \eta x; \eta\omega)\right\}^{\beta-1}, \quad x > 0. \quad (7)$$

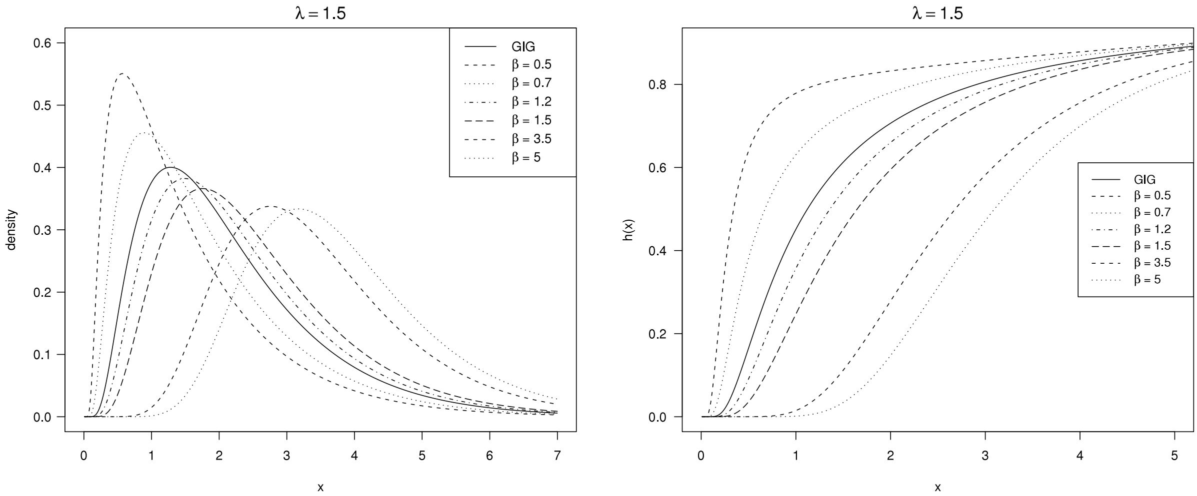

Fig. 1. Plots of the density (7) and hazard rate function (8) for some parameter values with $\eta = \omega = 1$.

Alternatively, we can rewrite $G(x)$ and $g(x)$, respectively, as $G(x) = C^{\beta}\eta^{-\lambda\beta}\gamma(\lambda, \eta x; \eta\omega)^{\beta}$ and $g(x) = \beta C^{\beta}\eta^{-\lambda(\beta-1)} x^{\lambda-1}\exp\{-(\eta x+\omega x^{-1})\}\gamma(\lambda, \eta x; \eta\omega)^{\beta-1}$. When β is equal to a positive integer value, n say, then $g(x)$ is the density function of the maximum statistic in a random sample of size n from the distribution $F(x)$. The survival $S(x)$ and hazard rate $h(x)$ are $S(x) = 1 - \{1 - C\eta^{-\lambda}\Gamma(\lambda, \eta x; \eta\omega)\}^{\beta}$ and

$$h(x) = \frac{\beta C x^{\lambda-1}\exp\{-(\eta x + \omega x^{-1})\}[1 - C\eta^{-\lambda}\Gamma(\lambda, \eta x; \eta\omega)]^{\beta-1}}{1 - \{1 - C\eta^{-\lambda}\Gamma(\lambda, \eta x; \eta\omega)\}^{\beta}}, \quad x > 0. \tag{8}$$

The EGIG distribution generalizes a few known distributions. The GIG distribution arises as a special sub-model for $\beta = 1$ and its special cases are also special sub-models of the EGIG distribution. In addition, an interesting particular case from the EGIG distribution arises for $\omega = 0$. When $z \to 0$, it is possible to show that $K_{\lambda}(z) = 2^{\lambda-1}z^{-\lambda}\Gamma(\lambda)$ (see Jørgensen, 1982). Thus, the normalizing constant C reduces to $C = \eta^{\lambda}/\Gamma(\lambda)$ for $\omega = 0$ and a new cdf is obtained as

$$G(x) = \left\{\frac{\gamma(\lambda, \eta x)}{\Gamma(\lambda)}\right\}^{\beta}, \quad x > 0, \tag{9}$$

which is referred to as the exponentiated gamma (EGamma) distribution. The density function corresponding to (9) has the form

$$g(x) = \frac{\beta\eta^{\lambda}x^{\lambda-1}\exp(-\eta x)}{\Gamma(\lambda)}\left\{\frac{\gamma(\lambda, \eta x)}{\Gamma(\lambda)}\right\}^{\beta-1}, \quad x > 0.$$

The exponentiated standard gamma (ESGamma) distribution (Nadarajah and Kotz, 2006) is a special case of the EGamma distribution (9) for $\eta = 1$ and, consequently, a special case of the EGIG distribution for $\omega = 0$ and $\eta = 1$. The mathematical properties derived in this note for the EGIG distribution hold for the EGamma and ESGamma distributions by setting $\omega = 0$ and $\omega = 0$ and $\eta = 1$, respectively. This fact happens for other special sub-models. It is evident that the EGIG distribution is much more flexible than the GIG distribution. Plots of the density function (7) and hazard rate function (8) for selected parameter values are shown in Fig. 1.

3. Quantile function

We provide power series expansions for the quantile functions of the GIG and EGIG distributions. First, we consider the GIG quantile function defined by $x = Q_{\mathrm{GIG}}(u) = F^{-1}(u)$, where $u \in (0, 1)$. Defining the set $I_i = \{(k, j); k - j = i\}$ for $i = 0, 1, \ldots$, Eq. (5) can be written as $F(x) = 1 - \rho - x^{\lambda}\sum_{i=0}^{\infty} b_i x^i$, where $b_i = \sum_{(k,j)\in I_i} a_{j,k}$. We can expand x^{λ} in a Taylor series to obtain $x^{\lambda} = \sum_{k=0}^{\infty}(\lambda)_k(x-1)^k/k! = \sum_{j=0}^{\infty} f_j x^j$, where $f_j = \sum_{k=j}^{\infty}(-1)^{k-j}\binom{k}{j}(\lambda)_k/k!$ and $(\lambda)_k = \lambda(\lambda-1)\ldots(\lambda-k+1)$ is the descending factorial. We have

$$F(x) = 1 - \rho - \left(\sum_{j=0}^{\infty} f_j x^j\right)\left(\sum_{i=0}^{\infty} b_i x^i\right).$$

By multiplying the two power series, we obtain $F(x) = 1 - \rho - \sum_{i=0}^{\infty} c_i x^i$, where $c_i = \sum_{k=0}^{i} f_k b_{i-k}$ (for $i = 0, 1, \ldots$). Then, we can rewrite $w = F(x)$ as

$$w = F(x) = w_0 - \sum_{i=1}^{\infty} c_i x^i, \qquad F'(x) = -c_1 \neq 0, \tag{10}$$

where $w_0 = 1 - \rho - c_0$. By inverting (10), the quantile function $x = Q_{\text{GIG}}(w) = F^{-1}(w)$ can be written as a power series expansion $x = Q_{\text{GIG}}(w) = \sum_{i=1}^{\infty} g_i (w - w_0)^i$, where $g_n = (1/n!)\mathrm{d}^{n-1}\Psi(x)^n/\mathrm{d}x^{n-1}|_{x=x_0}$ and $\Psi(x) = x/\{F(x) - w_0\}$. We have $\Psi(x) = -x/\sum_{i=1}^{\infty} c_i x^i = -1/\sum_{i=0}^{\infty} c_{i+1} x^i = \sum_{i=0}^{\infty} d_i x^i$. Here, the inverse of the power series follows from Gradshteyn and Ryzhik (2007). The coefficients d_i can be calculated recursively from the quantities c_i by $d_0 = 1$ and $d_i = c_1^{-1} \sum_{k=1}^{i} d_{i-k} c_{k+1}$ $(i \geq 1)$. We can obtain $\Psi(x)^n$ and then using a result due to Gradshteyn and Ryzhik (2007) for a power series raised to a positive integer n, it follows that

$$\Psi(x)^n = \left(\sum_{i=0}^{\infty} d_i x^i\right)^n = \sum_{i=0}^{\infty} q_{i,n} x^i, \quad n \geq 1, \tag{11}$$

where the coefficients $q_{i,n}$ (for $i = 1, 2, \ldots$) can be determined from the recurrence relation

$$q_{i,n} = i^{-1} \sum_{m=1}^{i} (nm - i + m) d_m q_{i-m,n} \quad \text{and} \quad q_{0,n} = 1. \tag{12}$$

The coefficient $q_{i,n}$ can be calculated from the quantities $q_{0,n} = 1, \ldots, q_{i-1,n}$. Clearly, $q_{i,n}$ can be given explicitly in terms of the coefficients d_i, but it is not necessary for programming numerically our expansions in any algebraic or numerical software. The power series with the first $(n+1)$ terms is $\Psi(x)^n = q_{0,n} + q_{1,n}x + \cdots + q_{n-1,n}x^{n-1} + q_{n,n}x^n + \cdots$. Simple differentiation gives $g_n = (1/n!)\mathrm{d}^{n-1}\Psi(x)^n/\mathrm{d}x^{n-1}|_{x=x_0} = q_{n-1,n}/n$ and, therefore, the quantile function $x = Q(w)$ reduces to

$$x = Q_{\text{GIG}}(w) = \sum_{n=1}^{\infty} \frac{q_{n-1,n}}{n} (w - w_0)^n, \tag{13}$$

where the coefficients $q_{i,n}$ are calculated from (12). They are basically implicit functions of the quantities b_i and f_i (for $i = 0, 1, \ldots$) defined above. Eq. (13) is the main result of this section.

We now derive an expansion for the EGIG quantile function $x = Q_{\text{EGIG}}(u) = G^{-1}(u)$. From (10), we can write $G(x) = F(x)^\beta = \left(1 - \rho - \sum_{i=0}^{\infty} c_i x^i\right)^\beta$, and then, after some algebra, we obtain

$$G(x) = \sum_{k=0}^{\infty} (-1)^k \binom{\beta}{k} \sum_{r=0}^{k} \binom{k}{r} \rho^{k-r} \left(\sum_{i=0}^{\infty} c_i x^i\right)^r = \sum_{i=0}^{\infty} h_i x^i, \tag{14}$$

where $h_i = \sum_{k=0}^{\infty} (-1)^k \binom{\beta}{k} \sum_{r=0}^{k} \binom{k}{r} \rho^{k-r} e_{i,r}$ $(i \geq 0)$, and the quantities $e_{i,r}$ (for $i = 1, 2, \ldots$) can be calculated from the recurrence relation

$$e_{i,r} = i^{-1} \sum_{m=1}^{i} (rm - i + m) c_m e_{i-m,r} \quad \text{and} \quad e_{0,r} = 1.$$

Hence, $G(x)$ in (14) has the form (10) and then Eq. (13) can be applied to obtain the EGIG quantile expansion by setting $w_0 = h_0$ and $c_i = -h_i$ (for $i \geq 1$) in (10).

Eq. (14) (and others expansions in this article) can be computed numerically in software such as MAPLE (Garvan, 2002), MATLAB (Sigmon and Davis, 2002) and MATHEMATICA (Wolfram, 2003). These symbolic software have currently the ability to deal with analytic expressions of formidable size and complexity.

4. Mixture form, moments and order statistics

From now on, let $X \sim \text{EGIG}(\lambda, \omega, \eta, \beta)$. We can write from (10)

$$[1 - F(x)]^{\beta-1} = \sum_{j=0}^{\infty} v_j x^j, \tag{15}$$

where $v_j = \sum_{i=0}^{\infty} (-1)^i \binom{\beta-1}{i} \sum_{r=0}^{i} \binom{i}{r} \rho^{i-r} t_{j,r}$ and the constants $t_{j,r}$ are determined by the recurrence relation $t_{j,r} = j^{-1} \sum_{m=1}^{j} (rm - j + m) c_m t_{j-m,r}$ (for $j \geq 1$) and $t_{0,r} = 1$. For evaluating v_j, the index i can stop at a very large number. If β is an integer, the index j in the above sum stops at $\beta-1$. By (7), we obtain $g(x) = \beta C(\lambda,\omega,\eta)\sum_{j=0}^{\infty} p_j v_j x^{\lambda+j-1}\exp\{-(\eta x+\omega x^{-1})\}$ and then

$$g(x) = \sum_{j=0}^{\infty} p_j f_j(x), \tag{16}$$

where $p_j = \beta v_j C(\lambda,\omega,\eta)/C(\lambda+j,\omega,\eta)$ and $f_j(x)$ denotes the density function of the GIG$(\lambda+j,\omega,\eta)$ distribution. Hence, the EGIG density function is an infinite linear combination of GIG densities and several of its mathematical properties can be obtained directly from those GIG properties.

The rth moment about zero and the mgf of X can be determined from (2) and (3) as

$$\mu'_r = \mathbb{E}(X^r) = \left(\frac{\omega}{\eta}\right)^{r/2}\sum_{j=0}^{\infty} p_j \frac{K_{\lambda+r+j}(2\sqrt{\eta\omega})}{K_{\lambda+j}(2\sqrt{\eta\omega})} \tag{17}$$

and

$$M(t) = \left(\frac{\eta}{\eta-t}\right)^{\lambda/2}\sum_{j=0}^{\infty} p_j\left(\frac{\eta}{\eta-t}\right)^{j/2}\frac{K_{\lambda+j}(2\sqrt{(\eta-t)\omega})}{K_{\lambda+j}(2\sqrt{\omega\eta})},$$

respectively. The nth descending factorial moment of X is given by $\mu'_{(n)} = \mathbb{E}\{X^{(n)}\} = \mathbb{E}\{X(X-1)(X-2)\cdots(X-n+1)\} = \sum_{r=0}^{n} s(n,r)\mu'_r$, where $s(n,r) = (r!)^{-1}[D^r x^{(n)}]_{x=0}$ are the Stirling numbers of the first kind. They count the number of ways to permute a list of n items into r cycles. Then, the factorial moments of X are obtained from (17). The central moments and cumulants (κ_s) of X follow immediately from the ordinary moments using well-known relationships. The skewness $\gamma_1 = \kappa_3/\kappa_2^{3/2}$ and kurtosis $\gamma_2 = \kappa_4/\kappa_2^2$ can be calculated from the second, third and fourth cumulants given by $\kappa_2 = \mu'_2 - \mu'^2_1$, $\kappa_3 = \mu'_3 - 3\mu'_2\mu'_1 + 2\mu'^3_1$ and $\kappa_4 = \mu'_4 - 4\mu'_3\mu'_1 - 3\mu'^2_2 + 12\mu'_2\mu'^2_1 - 6\mu'^4_1$.

The density of the ith order statistic $X_{i:n}$, say $g_{i:n}(x)$, in a random sample of size n from the EGIG distribution is $g_{i:n}(x) = g(x)G^{i-1}(x)\{1-G(x)\}^{n-i}/B(i,n+1-i)$ $(i=1,\ldots,n)$, where $B(\cdot,\cdot)$ is the beta function. Using (6) and (7), we can rewrite $g_{i:n}(x)$ as

$$g_{i:n}(x) = \frac{\beta C x^{\lambda-1}\exp\{-(\eta x+\omega x^{-1})\}}{B(i,n+1-i)}\,\frac{\left[1-\{1-C\eta^{-\lambda}\Gamma(\lambda,\eta x;\eta\omega)\}^{1-\beta}\right]^{n-i}}{\{1-C\eta^{-\lambda}\Gamma(\lambda,\eta x;\eta\omega)\}^{1-\beta i}}.$$

From the binomial expansion and the above results, we have

$$g_{i:n}(x) = \sum_{k=0}^{n-i}\frac{(-1)^k\binom{n-i}{k}}{B(i,n-i+1)}\left(\sum_{m,j=0}^{\infty} v_{j,k}^{(i)} p_m x^j f_m(x)\right),$$

where $v_{j,k}^{(i)} = \sum_{s=0}^{\infty}(-1)^s\binom{\beta(i+k-1)}{s}\sum_{r=0}^{s}\binom{s}{r}\rho^{s-r}t_{j,r}$. Hence, we write

$$g_{i:n}(x) = \sum_{k=0}^{n-i}\frac{(-1)^k\binom{n-i}{k}}{B(i,n-i+1)}\left(\sum_{m,j=0}^{\infty} s_{m,j,k}^{(i)} f_{m+j}(x)\right), \tag{18}$$

where $s_{m,j,k}^{(i)} = \beta v_{j,k}^{(i)} C(\lambda,\omega,\eta)/C(\lambda+m+j,\omega,\eta)$. The moments of $X_{i:n}$ can be expressed as

$$\mathbb{E}(X_{i:n}^r) = \left(\frac{\omega}{\eta}\right)^{r/2}\sum_{k=0}^{n-i}\frac{(-1)^k\binom{n-i}{k}}{B(i,n-i+1)}\left(\sum_{m,j=0}^{\infty} s_{m,j,k}^{(i)}\frac{K_{\lambda+m+j+r}(2\sqrt{\eta\omega})}{K_{\lambda+m+j}(2\sqrt{\eta\omega})}\right). \tag{19}$$

An alternative expression to (19) can be derived from the probability weighted moments (PWMs) (see Greenwood et al., 1979) using a result due to Barakat and Abdelkader (2004). We have $\mathbb{E}(X_{i:n}^r) = r\sum_{k=n-i+1}^{n}(-1)^{k-n+i-1}\binom{k-1}{n-i}\binom{n}{k}I_{r,k}$, where $I_{r,k} = \int_0^{\infty} x^{r-1}\{1-G(x)\}^k dx$. By expanding the binomial, we obtain $I_{r,k} = \sum_{m=0}^{k}\binom{k}{m}(-1)^m\tau_{r-1,m}$. Here, $\tau_{r,s} = \mathbb{E}\{X^r G(X)^s\} = \sum_{k,i=0}^{\infty}\binom{\beta s}{k}(-1)^k\sum_{r=0}^{k}\binom{k}{r}\rho^{k-r}h_{i,r}\mu'_{r+i}$, where $h_{i,r}$ can be determined recursively from $h_{i,r} = i^{-1}\sum_{m=1}^{i}(rm-i+m)c_m h_{i-m,r}$, (for $i=1,2,\ldots$), $h_{0,r}=1$ and μ'_{r+i} follows from (17).

The mgf of the ith order statistic, say $M_{i:n}(t)$, can be written from (3) and (18) as

$$M_{i:n}(t) = \sum_{k=0}^{n-i}\frac{(-1)^k\binom{n-i}{k}}{B(i,n-i+1)}\left\{\sum_{m,j=0}^{\infty} s_{m,j,k}^{(i)}\left(\frac{\eta}{\eta-t}\right)^{(\lambda+m+j)/2}\frac{K_{\lambda+m+j}(2\sqrt{(\eta-t)\omega})}{K_{\lambda+m+j}(2\sqrt{\omega\eta})}\right\}.$$

The L-moments (Hosking, 1990) are analogous to the ordinary moments but can be estimated by linear combinations of order statistics. They exist whenever the mean of the distribution exists, even though some higher moments may not exist, and are relatively robust to the effects of outliers. The rth L-moment is defined by $\tau_r = \sum_{j=0}^{r-1}(-1)^{r-1-j}\binom{r-1}{j}\binom{r-1+j}{j}\xi_j$, where $\xi_j = \mathbb{E}\{XG(X)^j\} = (j+1)^{-1}\mathbb{E}(X_{j+1:j+1})$. In particular, $\tau_1 = \xi_0$, $\tau_2 = 2\xi_1 - \xi_0$, $\tau_3 = 6\xi_2 - 6\xi_1 + \xi_0$ and $\tau_4 = 20\xi_3 - 30\xi_2 + 12\xi_1 - \xi_0$. The L-moments of the EGIG distribution can be obtained from (19).

5. Mean deviations and Rényi entropy

The amount of scatter in a population is evidently measured to some extent by the totality of deviations from the mean and median. If X has the EGIG distribution, we can derive the mean deviations about the mean $\mu_1' = \mathbb{E}(X)$ and about the median m from the density function (7) by

$$\delta_1 = \int_0^\infty |x - \mu_1'| g(x)\mathrm{d}x \quad \text{and} \quad \delta_2 = \int_0^\infty |x - m| g(x)\mathrm{d}x,$$

respectively. The mean μ_1' is calculated from (17) and the median m is the solution of the nonlinear equation $\Gamma(\lambda, \eta m; \eta\omega) = \eta^\lambda(1 - 2^{-1/\beta})/C$. These measures can be calculated from the relationships

$$\delta_1 = 2\left\{\mu_1' G(\mu_1') - J(\mu_1')\right\} \quad \text{and} \quad \delta_2 = \mu_1' - 2J(m), \tag{20}$$

where $J(p) = \int_0^p x g(x)\mathrm{d}x$ and $G(\mu_1')$ comes from (6). We have $J(p) = \eta^{-(\lambda+1)}\gamma(\lambda+1, \eta p; \eta\omega)\sum_{j=0}^\infty p_j C(\lambda+j, w, \eta)$, where p_j was defined in Section 4. Bonferroni and Lorenz curves have applications not only in economics to study income and poverty, but also in other fields like reliability, demography, insurance and medicine. For the EGIG distribution, these measures are given by $B(p) = J(q)/(p\mu_1')$ and $L(p) = J(q)/\mu_1'$, respectively, where $\mu_1' = \mathbb{E}(X)$ and $q = G^{-1}(p; \lambda, \eta, \omega)$ can be calculated for a given probability $p \in (0, 1)$ by solving the non-linear equation $\Gamma(\lambda, \eta q; \eta\omega) = \eta^\lambda(1 - p^{1/\beta})/C$.

The entropy of a random variable is a measure of uncertainty variation (see Song, 2001). The Rényi entropy is defined by $I_R(\delta) = (1-\delta)^{-1}\log\{\int_{\mathbb{R}} g^\delta(x)\mathrm{d}x\}$, where $\delta > 0$ and $\delta \neq 1$. We can write $g(x)^\delta = \beta^\delta C^\delta x^{\delta(\lambda-1)}\exp\{-(\delta\eta x + \delta\omega x^{-1})\}\sum_{i=0}^\infty h_i^\star x^i$, where $h_i^\star = \sum_{k=0}^\infty(-1)^k\binom{\delta(\beta-1)}{k}\sum_{r=0}^k\binom{k}{r}\rho^{k-r}e_{i,r}$ $(i \geq 0)$, and the quantities $e_{i,r}$ (for $i = 1, 2, \ldots$) can be determined from the c_i's by the recurrence relation given at the end of Section 3. Hence, the Rényi entropy reduces to

$$I_R(\delta) = (1-\delta)^{-1}\log\left\{\beta^\delta C(\lambda, \omega, \eta)^\delta \sum_{i=0}^\infty \frac{h_i^\star}{C(\delta(\lambda-1)+i+1, \omega, \eta)}\right\}.$$

6. Reliability

In the context of reliability, the stress–strength model describes the life of a component which has a random strength X_1 that is subjected to a random stress X_2. The component fails at the instant that the stress applied to it exceeds the strength, and the component will function satisfactorily whenever $X_1 > X_2$. Hence, $R = \Pr(X_2 < X_1)$ is a measure of component reliability. It has wide applications in the engineering area. First, we derive the reliability R when X_1 and X_2 have independent $\mathrm{GIG}(\lambda_1, \omega_1, \eta_1)$ and $\mathrm{GIG}(\lambda_2, \omega_2, \eta_2)$ distributions with different sets of parameters. We readily obtain from (5)

$$R = \int_0^\infty f_1(x)F_2(x)\mathrm{d}x = 1 - \rho_2 - \sum_{j,k=0}^\infty a_{j,k}^{(2)}\mathbb{E}(X_1^{k-j+\lambda_2}), \tag{21}$$

where $\rho_2 = \rho(\lambda_2, \eta_2, \omega_2) = C_2\eta_2^{-\lambda_2}\sum_{j=0}^\infty\Gamma(\lambda_2 - j)(-\eta_2\omega_2)^j/j!$, $a_{j,k}^{(2)} = (-1)^{k+j+1}C_2\eta_2^k\omega_2^j/\{(k-j+\lambda_2)j!k!\}$ and $\mathbb{E}(X_1^{k-j+\lambda_2}) = (\omega_1/\eta_1)^{(k-j+\lambda_2)/2}K_{k-j+\lambda_1+\lambda_2}\big(2\sqrt{\eta_1\omega_1}\big)/K_{\lambda_1}\big(2\sqrt{\eta_1\omega_1}\big)$. The reliability R^* when Y_1 and Y_2 have independent $\mathrm{EGIG}(\lambda_1, \omega_1, \eta_1, \beta_1)$ and $\mathrm{EGIG}(\lambda_2, \omega_2, \eta_2, \beta_2)$ distributions can be derived as follows. From Eq. (16), we have

$$R^* = \int_0^\infty g_1(x)G_2(x)\mathrm{d}x = \sum_{j,k=0}^\infty p_{(1)j}p_{(2)k}\int_0^\infty f_{(1)j}(x)F_{(2)k}(x)\mathrm{d}x,$$

where $f_{(r)j}(x)$ and $F_{(r)j}(x)$ denote the density and cumulative functions of the $\mathrm{GIG}(\lambda_r + j, \omega_r, \eta_r)$ distribution (for $r = 1, 2$) and the $p_{(r)j}$ are the coefficients of the corresponding linear combination form. Eq. (1) yields $f_{(1)j}(x) = C(\lambda_1 + j, \omega_1, \eta_1)x^{\lambda_1+j-1}\exp\{-(\eta_1 x + \omega_1 x^{-1})\}$ and (10) gives $G_{(2)k}(x) = 1 - \rho_2^{(k)} - \sum_{i=0}^\infty c_{(2)i}^{(k)}x^i$, where $\rho_2^{(k)} = \rho(\lambda_2 + k, \eta_2, \omega_2) = C(\lambda_2+k, \omega_2, \eta_2)\eta^{-\lambda_2-k}\sum_{r=0}^\infty\Gamma(\lambda_2+k-r)(-\eta_2\omega_2)^r/r!$, $c_{(2)i}^{(k)} = \sum_{m=0}^i f_{(2)m}^{(k)}b_{(2)(i-m)}^{(k)}$, $f_{(2)m}^{(k)} = \sum_{p=m}^\infty(-1)^{p-m}\binom{p}{m}(\lambda_2+k)_p/p!$, $b_{(2)i}^{(k)} = \sum_{(p,j)\in I_i}a_{(2)(j,p)}^{(k)}$, $a_{(2)(j,p)}^{(k)} = (-1)^{p+j+1}C(\lambda_2+k, \eta_2, \omega_2)\eta_2^p\omega_2^j/\{(p-j+\lambda)j!p!\}$ (for $i = 0, 1, \ldots$), $I_p = \{(k, j); k - j = p\}$ and $p \geq 0$. The coefficients in R^* are determined by $p_{(r)j} = C(\lambda_r, \omega_r, \eta_r)\beta_r v_{(r)j}/C(\lambda_r + j, \omega_r, \eta_r)$ for $r = 1, 2$. Here,

$v_{(r)j} = \sum_{i=0}^{\infty}(-1)^i \binom{\beta_r - 1}{i} \sum_{p=0}^{i} \binom{i}{p} \rho_r^{i-p} t_{(r)j,p}$, $t_{(r)j,p}$ is calculated by the recurrence relation $t_{(r)j,p} = j^{-1}\sum_{m=1}^{j}(pm - j + m)c_{(r)m}t_{(r)j-m,p}$ (for $j \geq 1$) and $t_{(r)0,p} = 1$, where $c_{(r)i}$ are the coefficients in the expansion of the GIG$(\lambda_r, \omega_r, \eta_r)$ cumulative distribution (10). Thus,

$$\int_0^{\infty} f_j^{(1)}(x)F_k^{(2)}(x)\mathrm{d}x = 1 - \rho_2^{(k)} \sum_{i=0}^{\infty} c_{2i}^{(k)} \int_0^{\infty} C(\lambda_1 + j, \omega_1, \eta_1)x^{\lambda_1+i+j-1}\exp\{-(\eta_1 x + \omega_1 x^{-1})\}\mathrm{d}x.$$

This above integral is the ith moment of the GIG$(\lambda_1 + j, \omega_1, \eta_1)$ distribution and using (2), we obtain

$$R^* = 1 - \sum_{k=0}^{\infty} p_{2k}\rho_2^{(k)} - \sum_{j,k,i=0}^{\infty} p_{(1)j}p_{(2)k}c_{(2)i}^{(k)}\left(\frac{\omega_1}{\eta_1}\right)^{i/2}\frac{K_{\lambda_1+i+j}(2\sqrt{\eta_1\omega_1})}{K_{\lambda_1+j}(2\sqrt{\eta_1\omega_1})}.$$

7. Estimation

We consider estimation of the model parameters by the method of maximum likelihood. Let $\boldsymbol{x} = (x_1, \ldots, x_n)^{\top}$ denote n independent observations from the EGIG distribution with unknown parameter vector $\boldsymbol{\theta} = (\lambda, \eta, \omega, \beta)^{\top}$. The total log-likelihood function for $\boldsymbol{\theta}$ can be written as

$$\ell(\boldsymbol{\theta}) = \ell(\lambda, \eta, \omega, \beta) = n\log(\beta C) + (\lambda - 1)\sum_{i=1}^{n}\log(x_i) - \sum_{i=1}^{n}(\eta x_i + \omega x_i^{-1})$$
$$+ (\beta - 1)\sum_{i=1}^{n}\log\{1 - C\eta^{-\lambda}\Gamma(\lambda, \eta x_i; \eta\omega)\}. \tag{22}$$

For maximizing the log-likelihood (22), we consider λ fixed and then obtain the likelihood equations for estimating η, ω and β. In principle, λ could also be estimated directly by maximum likelihood but there may be practical difficulties, since in general the modified Bessel function, and hence the total log-likelihood, is difficult to handle as a function of the parameter λ, and then the derivatives of $\ell(\boldsymbol{\theta})$ with respect to λ seem very difficult. Further, the generalized incomplete gamma function also depends on λ and may also present numerical problems. Thus, potential computational problems can be avoided using an indirect method to estimate λ. Jørgensen (1982) also used an indirect method to estimate λ for the GIG distribution. Here, we proceed in this direction.

For λ fixed, the function (22) can be written as $\ell^{(\lambda)} = \ell^{(\lambda)}(\eta, \omega, \beta)$. Thus, the score vector $\boldsymbol{U}^{(\lambda)}(\eta, \omega, \beta) = (\partial\ell^{(\lambda)}/\partial\eta, \partial\ell^{(\lambda)}/\partial\omega, \partial\ell^{(\lambda)}/\partial\beta)^{\top}$ has components

$$\frac{\partial\ell^{(\lambda)}}{\partial\eta} = \frac{n\omega^{1/2}R_\lambda}{\eta^{1/2}} - \sum_{i=1}^{n}x_i + C\eta^{-\lambda}(\beta - 1)\sum_{i=1}^{n}\frac{\{\lambda/\eta - (\omega/\eta)^{1/2}R_\lambda\}\Gamma_{i,0} + S_i}{1 - C\eta^{-\lambda}\Gamma_{i,0}},$$
$$\frac{\partial\ell^{(\lambda)}}{\partial\omega} = -\frac{n\lambda}{\omega} + \frac{n\eta^{1/2}R_\lambda}{\omega^{1/2}} - \sum_{i=1}^{n}\frac{1}{x_i} + C\eta^{-\lambda}(\beta - 1)\sum_{i=1}^{n}\frac{\{\lambda/\omega - (\eta/\omega)^{1/2}R_\lambda\}\Gamma_{i,0} + \eta\Gamma_{i,1}}{1 - C\eta^{-\lambda}\Gamma_{i,0}},$$
$$\frac{\partial\ell^{(\lambda)}}{\partial\beta} = \frac{n}{\beta} + \sum_{i=1}^{n}\log\{1 - C\eta^{-1}\Gamma_{i,0}\},$$

where $R_\lambda = R_\lambda(\eta, \omega) = K_{\lambda+1}(2\sqrt{\eta\omega})/K_\lambda(2\sqrt{\eta\omega})$, $\Gamma_{i,j} = \Gamma(\lambda - j, \eta x_i; \eta\omega)$, $S_i = S_i(\lambda, \eta, \omega; x_i) = \omega\Gamma_{i,1} + \eta^{\lambda-1}x_i^{\lambda}\exp(-\eta x_i - \omega x_i^{-1})$, for $i = 1, \ldots, n$ and $j \in \{0, 1, 2\}$.

The SQP method (see Lawrence and Tits, 2001) with analytical derivatives has been used for maximizing the log-likelihood function $\ell(\boldsymbol{\theta})$ in order to obtain the restricted maximum likelihood estimates (MLEs) $\widehat{\eta}^{(\lambda)}$, $\widehat{\omega}^{(\lambda)}$ and $\widehat{\beta}^{(\lambda)}$ of η, ω and β, respectively. This method is implemented in the Ox matrix programming language (Doornik, 2006) through the subroutine MaxSQPF. We add the superscript (λ) to emphasize the dependence of the estimates on this parameter.

From $\partial\ell^{(\lambda)}/\partial\beta = 0$, we immediately obtain

$$\widehat{\beta}^{(\lambda)} = -n\Big/\sum_{i=1}^{n}\log\{1 - \widehat{C}^{(\lambda)}\widehat{\Gamma}_{i,0}^{(\lambda)}/(\widehat{\eta}^{(\lambda)})^{\lambda}\},$$

where $\widehat{C}^{(\lambda)} = C(\lambda, \widehat{\omega}^{(\lambda)}, \widehat{\eta}^{(\lambda)}) = (\widehat{\eta}^{(\lambda)}/\widehat{\omega}^{(\lambda)})^{\lambda/2}/\{2K_\lambda(2\sqrt{\widehat{\eta}^{(\lambda)}\widehat{\omega}^{(\lambda)}})\}$ and $\widehat{\Gamma}_{i,0}^{(\lambda)} = \Gamma(\lambda, \widehat{\eta}^{(\lambda)}x_i; \widehat{\eta}^{(\lambda)}\widehat{\omega}^{(\lambda)})$ for $i = 1, \ldots, n$. By replacing in $\ell(\boldsymbol{\theta})$, η, ω and β by $\widehat{\eta}^{(\lambda)}$, $\widehat{\omega}^{(\lambda)}$ and $\widehat{\beta}^{(\lambda)}$, respectively, we obtain the profile log-likelihood function for λ as

$$\ell_p(\lambda) = n\log\{\widehat{\beta}^{(\lambda)}\widehat{C}^{(\lambda)}\} + (\lambda - 1)\sum_{i=1}^{n}\log(x_i) - \sum_{i=1}^{n}(\widehat{\eta}^{(\lambda)}x_i + \widehat{\omega}^{(\lambda)}x_i^{-1})$$
$$+ (\widehat{\beta}^{(\lambda)} - 1)\sum_{i=1}^{n}\log\left\{1 - \widehat{C}^{(\lambda)}\widehat{\Gamma}_{i,0}^{(\lambda)}/(\widehat{\eta}^{(\lambda)})^{\lambda}\right\}. \tag{23}$$

Table 1
Estimates and log-likelihood functions.

Distribution	Estimates				$\ell(\widehat{\boldsymbol{\theta}})$
	λ	η	ω	β	
EGIG	34.860	8.399	1.08×10^{-14}	0.127	−141.72
	(–)	(0.4681)	(0.0218)	(0.0172)	
EGamma	34.860	8.399	0	0.127	−141.72
	(0.9838)	(0.4662)		(0.0185)	
GIG	5.953	2.271	2.52×10^{-16}	1	−143.23
	(4.215; 7.385)	(0.3265)	(0.7819)		
Gamma	5.953	2.271	0	1	−143.23
	(0.8193)	(0.3261)			
ESGamma	1.092	1	0	6.553	−146.15
	(0.2993)			(3.1302)	
Hyperbola	0	0.942	5.286	1	−149.96
		(0.1470)	(0.8179)		
Inverse Gaussian	−1/2	0.848	5.826	1	−150.73
		(0.1449)	(0.8239)		

The plot of the profile log-likelihood $\ell_p(\lambda)$ against λ for a trial series of values determines numerically the value of the estimate $\widehat{\lambda}$ which maximizes (23). We only need to find the value $\widehat{\lambda}$ such that

$$\widehat{\lambda} = \arg\max_{\lambda} \ell_p(\lambda).$$

Once $\widehat{\lambda}$ is obtained from the plot, it can be substituted in the log-likelihood $\ell(\boldsymbol{\theta})$ to produce the unrestricted MLEs $\widehat{\eta} = \widehat{\eta}^{(\widehat{\lambda})}$, $\widehat{\omega} = \widehat{\omega}^{(\widehat{\lambda})}$ and $\widehat{\beta} = \widehat{\beta}^{(\widehat{\lambda})}$.

We can use the following two-step algorithm for maximum likelihood estimation of λ, η, ω and β. (i) For a given λ, the log-likelihood function (22) is maximized with respect to η, ω and β, that is, we compute $\widehat{\eta}^{(\lambda)}$, $\widehat{\omega}^{(\lambda)}$ and $\widehat{\beta}^{(\lambda)}$. Thus, we compute $\ell_p(\lambda)$ from (23). (ii) Vary λ over a grid of values to obtain the estimate $\widehat{\lambda}$ that maximizes (23). In the next section, we shall adopt this algorithm to obtain approximate MLEs of the parameters λ, η, ω and β in model (7).

The observed information matrix for the parameters η, ω and β of the EGIG distribution is given in Appendix A. The log-likelihood function, score function and the observed information matrix for the unknown parameters of the EGamma distribution can be found in Appendix B.

8. Application

In this section, we compare the fits of the EGIG, EGamma, GIG, gamma, ESGamma, inverse Gaussian and hyperbola distributions to a real data set. All the computations were done using the Ox matrix programming language. Ox is freely distributed for academic purposes and available at http://www.doornik.com. For maximizing the log-likelihood functions, we use the subroutine `MaxSQPF` with analytical derivatives.

We shall consider an uncensored data set from Nichols and Padgett (2006) consisting of 100 observations on breaking stress of carbon fibres (in Gba). The data are: 0.39, 0.81, 0.85, 0.98, 1.08, 1.12, 1.17, 1.18, 1.22, 1.25, 1.36, 1.41, 1.47, 1.57, 1.57, 1.59, 1.59, 1.61, 1.61, 1.69, 1.69, 1.71, 1.73, 1.80, 1.84, 1.84, 1.87, 1.89, 1.92, 2.00, 2.03, 2.03, 2.05, 2.12, 2.17, 2.17, 2.17, 2.35, 2.38, 2.41, 2.43, 2.48, 2.48, 2.50, 2.53, 2.55, 2.55, 2.56, 2.59, 2.67, 2.73, 2.74, 2.76, 2.77, 2.79, 2.81, 2.81, 2.82, 2.83, 2.85, 2.87, 2.88, 2.93, 2.95, 2.96, 2.97, 2.97, 3.09, 3.11, 3.11, 3.15, 3.15, 3.19, 3.19, 3.22, 3.22, 3.27, 3.28, 3.31, 3.31, 3.33, 3.39, 3.39, 3.51, 3.56, 3.60, 3.65, 3.68, 3.68, 3.68, 3.70, 3.75, 4.20, 4.38, 4.42, 4.70, 4.90, 4.91, 5.08, 5.56.

Table 1 lists the MLEs of the parameters (standard errors between parentheses) and the values of the log-likelihood functions. We also give in this table an approximate 90% confidence interval for the parameter λ of the GIG distribution. An approximate 90% confidence interval for λ of the EGIG distribution was not obtained because there were numerical problems in the evaluation of the modified Bessel function for large values of λ. Note that the EGIG and EGamma distributions produce the highest value of the log-likelihood function. It is noteworthy that these distributions have the same estimates for the parameters λ, η and β. It is expected since $\widehat{\omega} \approx 0$ for the EGIG distribution. Recall that the EGamma distribution is a special case of the EGIG for $\omega = 0$. It also happens with the GIG and gamma distributions, as expected, since $\widehat{\omega} \approx 0$ for the GIG distribution.

The parameter λ for the EGIG and GIG distributions was estimated using the profile log-likelihood function. Fig. 2 shows the profile log-likelihood curve plotted against the parameter λ for the EGIG and GIG distributions. For the EGIG distribution, its maximum of −141.72 occurs near $\lambda = 34.860$, whereas for the GIG distribution its maximum of −143.23 occurs near $\lambda = 5.953$.

Plots of the estimated pdf and cdf of the fitted models are given in Fig. 3. It is clear that the EGIG and EGamma, and GIG and gamma distributions present the same fits, i.e. there is no difference between the curves. Additionally, these plots indicate that the EGIG and EGamma models provide better fits than the other models.

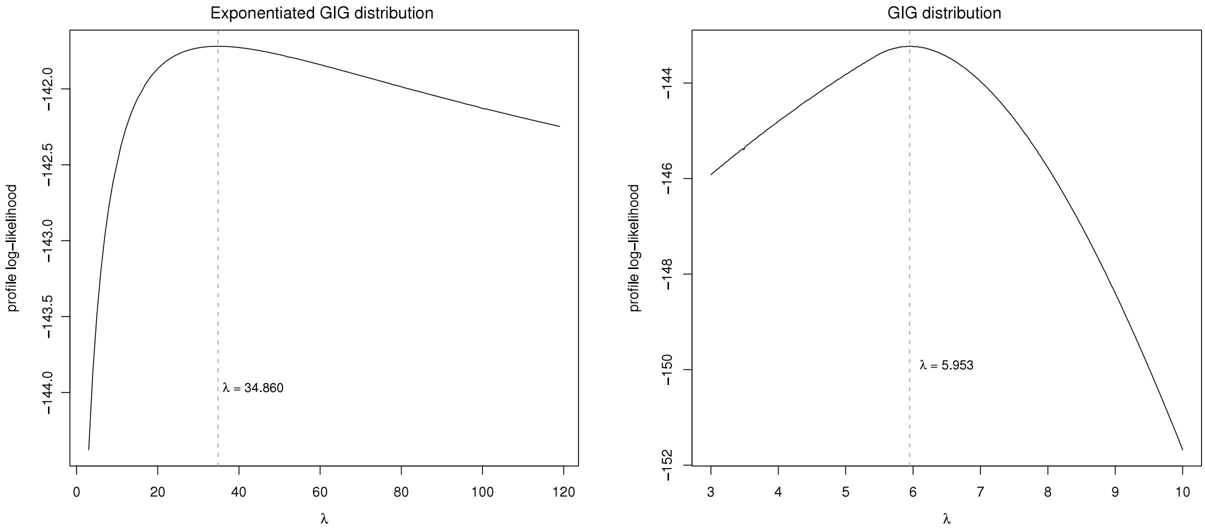

Fig. 2. The profile log-likelihood curves for λ for the EGIG and GIG models fitted to the data from Nichols and Padgett (2006).

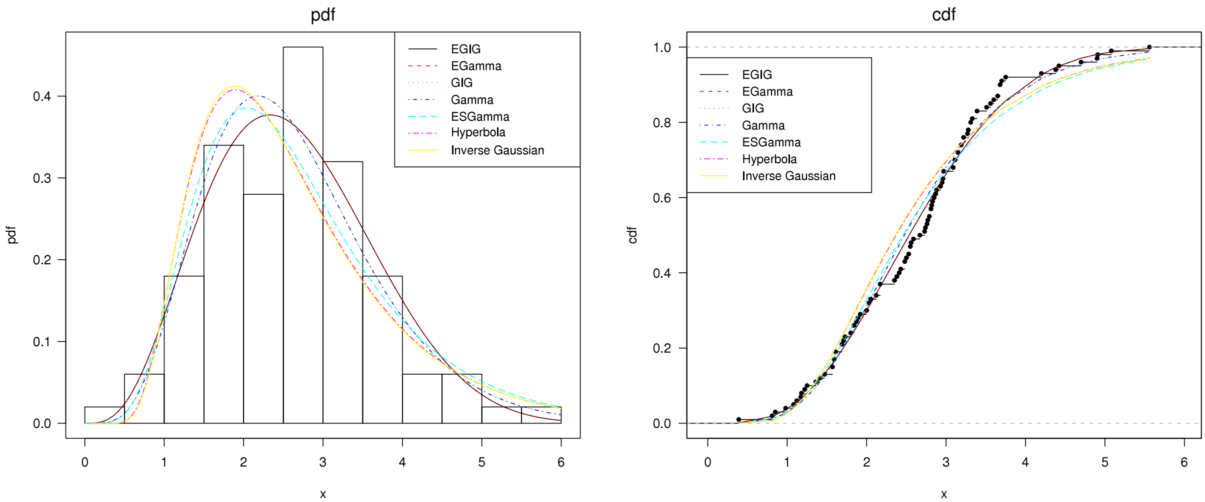

Fig. 3. Estimated pdf and cdf of the EGIG, EGamma, GIG, Gamma, ESGamma, Hyperbola and Inverse Gaussian distributions for the data from Nichols and Padgett (2006).

Table 2
Goodness-of-fit tests.

Distribution	Statistics	
	W^*	A^*
EGIG	0.08616	0.48662
EGamma	0.08616	0.48662
GIG	0.14802	0.75721
Gamma	0.14802	0.75721
ESGamma	0.21838	1.13910
Hyperbola	0.29658	1.60980
Inverse Gaussian	0.31370	1.71010

In what follows, we shall apply formal goodness-of-fit tests in order to verify which distribution fits better to the data from Nichols and Padgett (2006). We apply the Cramér–von Mises (W^*) and Anderson–Darling (A^*) test statistics. The W^* and A^* test statistics are described in details in Chen and Balakrishnan (1995). In general, the smaller the values of W^* and A^* statistics, the better the fit to the data. The values of these statistics for all models are given in Table 2. As expected, the values of W^* and A^* for the EGIG and EGamma, and GIG and gamma distributions are exactly the same. According to these statistics, the EGIG and EGamma distributions produce better fits to these data than the other distributions.

In summary, the proposed distributions EGIG and EGamma produce better fits for the data from Nichols and Padgett (2006) than other known distributions in the literature. In this case, the EGamma distribution could be chosen since it has fewer parameters to be estimated.

9. Conclusions

In this note, we propose the so-called exponentiated generalized inverse Gaussian (EGIG) distribution to extend several widely known distributions in the lifetime literature. It includes, as special sub-models, the generalized inverse Gaussian (GIG) distribution (Good, 1953) and the exponentiated standard gamma distribution (Nadarajah and Kotz, 2006). We provide a mathematical treatment of the new distribution including expansions for the density function, moments, moment generating function, mean deviations, reliability and Rényi entropy. We examine a maximum likelihood estimation of the model's parameters and derive the observed information matrix. An application of the new distribution to a real data set is given to demonstrate that it can be used quite effectively to provide better fits than other available models. We hope this generalization may attract wider applications in survival analysis.

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Appendix A. Observed information matrix for the EGIG distribution

The observed information matrix $\boldsymbol{J}^{(\lambda)}(\eta, \omega, \beta)$ is

$$\boldsymbol{J}^{(\lambda)}(\eta, \omega, \beta) = -\begin{pmatrix} U_{\eta\eta} & U_{\eta\omega} & U_{\eta\beta} \\ \cdot & U_{\omega\omega} & U_{\omega\beta} \\ \cdot & \cdot & U_{\beta\beta} \end{pmatrix},$$

whose elements are: $U_{\beta\beta} = -n/\beta^2$,

$$\begin{aligned} U_{\eta\eta} = & -\frac{n\lambda}{4\eta^2} - \frac{3n\lambda\omega D_\lambda}{2\eta\sqrt{\eta\omega}} - \frac{n\omega H_\lambda}{\eta} + \frac{n\omega\{\lambda + 2D_\lambda\}D_\lambda}{2\eta} - \frac{nC\omega^3 D_\lambda}{2(\eta\omega)^{3/2}} + \frac{nC\omega\{\lambda(\eta\omega)^{-1/2} - 2K_\lambda(2\sqrt{\eta\omega})D_\lambda^2\}}{\eta(\eta/\omega)^{\lambda/2}} \\ & - C\eta^{-\lambda}(\beta-1)\sum_{i=1}^{n}\frac{\{\lambda(\lambda+2)/(4\eta^2) + \lambda\omega^2/[4(\eta\omega)^{3/2}]\}\Gamma_{i,0} + \{\lambda/\eta + \omega D_\lambda/\sqrt{\eta\omega}\}S_i}{1 - C\eta^{-\lambda}\Gamma_{i,0}} \\ & - C\eta^{-\lambda}(\beta-1)\sum_{i=1}^{n}\frac{\omega D_\lambda\{\lambda/(2\eta\sqrt{\eta\omega}) + \omega/\{2(\eta\omega)^{3/2}\} + D_\lambda/\eta\}\Gamma_{i,0} - \omega H_\lambda\Gamma_{i,0}/\eta + \eta^{\lambda-1}\{I_i + T_i\}}{1 - C\eta^{-\lambda}\Gamma_{i,0}} \\ & - C^2\eta^{-2\lambda}(\beta-1)\sum_{i=1}^{n}\left\{\frac{\{\lambda/(2\eta) + \omega D_\lambda/\sqrt{\eta\omega}\}\Gamma_{i,0} + S_i}{1 - C\eta^{-\lambda}\Gamma_{i,0}}\right\}^2, \end{aligned}$$

$$\begin{aligned} U_{\eta\omega} = & \frac{n\lambda}{4\eta\omega} - \frac{nD_\lambda}{\sqrt{\eta\omega}} - nH_\lambda + C\eta^{-\lambda}(\beta-1)\sum_{i=1}^{n}\frac{\{H_\lambda - \lambda/2 - \lambda(\lambda+1)/(4\eta\omega) - 2D_\lambda + (1-\lambda)D_\lambda/(2\sqrt{\eta\omega})\}\Gamma_{i,0}}{1 - C\eta^{-\lambda}\Gamma_{i,0}} \\ & + nD_\lambda^2 - C\eta^{-\lambda}(\beta-1)\sum_{i=1}^{n}\frac{\{\lambda/2 + \sqrt{\eta\omega}D_\lambda\}\Gamma_{i,1} + \{\lambda/(2\omega) + \sqrt{\eta}D_\lambda/\sqrt{\omega}\}S_i}{1 - C\eta^{-\lambda}\Gamma_{i,0}} \\ & - C^2\eta^{-2\lambda}(\beta-1)\sum_{i=1}^{n}\frac{\left[\{\lambda/(2\eta) + \sqrt{\omega}D_\lambda/\sqrt{\eta}\}\Gamma_{i,0} + S_i\right]\left[\{\lambda/(2\omega) + \sqrt{\eta}D_\lambda/\sqrt{\omega}\}\Gamma_{i,0} - \eta\Gamma_{i,1}\right]}{(1 - C\eta^{-\lambda}\Gamma_{i,0})^2}, \end{aligned}$$

$$\begin{aligned} U_{\omega\omega} = & \frac{3n\lambda}{4\omega^2} - \frac{n\eta H_\lambda}{\omega} + \frac{n(1-\lambda)\eta^{1/2}D_\lambda}{2\omega^{3/2}} + \frac{n\eta D_\lambda}{\omega} - C\eta^{-\lambda}(\beta-1)\sum_{i=1}^{n}\frac{\{\lambda\eta/\omega + 2\eta^2 D_\lambda/\sqrt{\eta\omega}\}\Gamma_{i,1} + \eta^2\Gamma_{i,2}}{1 - C\eta^{-\lambda}\Gamma_{i,0}} \\ & + C\eta^{-\lambda}(\beta-1)\sum_{i=1}^{n}\frac{\{\eta H_\lambda/\omega - (\lambda+3)\lambda/(4\omega^2) - (1+2\lambda)\eta^{1/2}D_\lambda/(2\omega^{3/2}) - 2\eta D_\lambda^2/\omega\}\Gamma_{i,0}}{1 - C\eta^{-\lambda}\Gamma_{i,0}} \\ & - C^2\eta^{-2\lambda}(\beta-1)\sum_{i=1}^{n}\left\{\frac{\{\lambda/(2\omega) + \sqrt{\eta}D_\lambda/\sqrt{\omega}\}\Gamma_{i,0} + \eta\Gamma_{i,1}}{1 - C\eta^{-\lambda}\Gamma_{i,0}}\right\}^2, \end{aligned}$$

$$U_{\eta\beta} = C\eta^{-\lambda}\sum_{i=1}^{n}\frac{\{\lambda/(2\eta) + \sqrt{\omega}D_\lambda/\sqrt{\eta}\}\Gamma_{i,0} + S_i}{1 - C\eta^{-\lambda}\Gamma_{i,0}}, \qquad U_{\omega\beta} = C\eta^{-\lambda}\sum_{i=1}^{n}\frac{\{\lambda/(2\omega) + \sqrt{\eta}D_\lambda/\sqrt{\omega}\}\Gamma_{i,0} + \eta\Gamma_{i,1}}{1 - C\eta^{-\lambda}\Gamma_{i,0}},$$

where $D_\lambda = D_\lambda(\eta,\omega) = \lambda/\{2\sqrt{\eta\omega}\} - R_\lambda$, R_λ was defined in Section 7, $H_\lambda = H_\lambda(\eta,\omega) = 1 + (\lambda+1)R_\lambda/\{2\sqrt{\eta\omega}\}$, $I_i = I_i(\lambda,\eta,\omega;x_i) = \omega^2\Gamma_{i,2} + \eta^{\lambda-1}x_i^{\lambda+1}\exp(-\eta x_i - \omega x_i^{-1})$ and $T_i = T_i(\lambda,\eta,\omega;x_i) = \{\omega x_i^{-1} - (\lambda-1)\}\eta^{\lambda-2}x_i^{\lambda}\exp(-\eta x_i - \omega x_i^{-1})$, for $i = 1,\ldots,n$.

Under conditions that are fulfilled for parameters in the interior of the parameter space but not on the boundary, the asymptotic distribution of $(\widehat{\eta},\widehat{\omega},\widehat{\beta})^\top - (\eta,\omega,\beta)^\top$ can be approximated by $\mathcal{N}_3\big(\mathbf{0},\boldsymbol{J}(\widehat{\eta},\widehat{\omega},\widehat{\beta})^{-1}\big)$, where $\boldsymbol{J}(\widehat{\eta},\widehat{\omega},\widehat{\beta}) = \boldsymbol{J}^{(\widehat{\lambda})}(\widehat{\eta},\widehat{\omega},\widehat{\beta})$. Thus, the multivariate normal $\mathcal{N}_3\big(\mathbf{0},\boldsymbol{J}(\widehat{\eta},\widehat{\omega},\widehat{\beta})^{-1}\big)$ approximation can be used to construct confidence intervals and confidence regions for the parameters. In fact, asymptotic $100(1-\eta)\%$ confidence intervals for η, ω and β are given by $\widehat{\eta}\pm z_{\alpha/2}\times\{\widehat{\text{var}}(\widehat{\eta})\}^{1/2}$, $\widehat{\omega}\pm z_{\alpha/2}\times[\widehat{\text{var}}(\widehat{\omega})]^{1/2}$ and $\widehat{\beta}\pm z_{\alpha/2}\times[\widehat{\text{var}}(\widehat{\beta})]^{1/2}$, where $\text{var}(\cdot)$ is the diagonal element of $\boldsymbol{J}(\widehat{\eta},\widehat{\omega},\widehat{\beta})^{-1}$ corresponding to each parameter, and $z_{\alpha/2}$ is the quantile $(1-\alpha/2)$ of the standard normal distribution.

Appendix B. Estimation for the EGamma distribution

The log-likelihood function for the unknown parameter vector $\boldsymbol{\theta} = (\lambda,\eta,\beta)^\top$ of the EGamma distribution is given by

$$\ell = \ell(\boldsymbol{\theta}) = n\log(\beta) + n\lambda\log(\eta) - n\beta\log\{\Gamma(\lambda)\} + (\lambda-1)\sum_{i=1}^n\log(x_i) - \eta\sum_{i=1}^n x_i + (\beta-1)\sum_{i=1}^n\gamma(\lambda,\eta x_i).$$

The above log-likelihood function does not depend on the modified Bessel function. It implies that the derivatives with respect to the parameter λ can be obtained. The components of the score function are:

$$\frac{\partial\ell}{\partial\lambda} = n\log(\eta) - n\beta\psi(\lambda) + \sum_{i=1}^n\log(x_i) + (\beta-1)\sum_{i=1}^n\frac{\gamma'(\lambda,\eta x_i)}{\gamma(\lambda,\eta x_i)},$$

$$\frac{\partial\ell}{\partial\eta} = \frac{n\lambda}{\eta} - \sum_{i=1}^n x_i + (\beta-1)\eta^{\lambda-1}\sum_{i=1}^n\frac{x_i^\lambda\exp(-\eta x_i)}{\gamma(\lambda,\eta x_i)},\qquad \frac{\partial\ell}{\partial\beta} = \frac{n}{\beta} - n\log\{\Gamma(\lambda)\} + \sum_{i=1}^n\log\{\gamma(\lambda,\eta x_i)\},$$

where $\psi(\cdot)$ is the digamma function and $\gamma'(\lambda,\eta x_i) = \int_0^{\eta x_i}\log(t)t^{\lambda-1}\exp(-t)\mathrm{d}t$, for $i = 1,\ldots,n$. The maximization of ℓ can be done directly without the necessity of an indirect method as the case of the EGIG distribution. We use the SQP method with analytical derivatives (above presented) in order to obtain the MLEs $\widehat{\lambda}$, $\widehat{\eta}$ and $\widehat{\beta}$ of λ, η and β, respectively. The observed information matrix $\boldsymbol{J}(\boldsymbol{\theta})$ is

$$\boldsymbol{J}(\boldsymbol{\theta}) = -\begin{pmatrix} U_{\lambda\lambda} & U_{\lambda\eta} & U_{\lambda\beta}\\ \cdot & U_{\eta\eta} & U_{\eta\beta}\\ \cdot & \cdot & U_{\beta\beta}\end{pmatrix},$$

whose elements are

$$U_{\lambda\lambda} = -n\beta\psi'(\lambda) + (\beta-1)\sum_{i=1}^n\left\{\frac{\gamma''(\lambda,\eta x_i)}{\gamma(\lambda,\eta x_i)} - \left(\frac{\gamma'(\lambda,\eta x_i)}{\gamma(\lambda,\eta x_i)}\right)^2\right\},$$

$$U_{\lambda\eta} = \frac{n}{\eta} + (\beta-1)\eta^{\lambda-1}\sum_{i=1}^n\frac{x_i^\lambda\exp(-\eta x_i)\log(\eta x_i)}{\gamma(\lambda,\eta x_i)} - (\beta-1)\eta^{\lambda-1}\sum_{i=1}^n\frac{x_i^\lambda\exp(-\eta x_i)\gamma'(\lambda,\eta x_i)}{\gamma(\lambda,\eta x_i)^2},$$

$$U_{\eta\eta} = -\frac{n\lambda}{\eta^2} + (\beta-1)\eta^{\lambda-1}\sum_{i=1}^n\frac{x_i^\lambda\exp(-\eta x_i)\{\log(\eta) - x_i\}}{\gamma(\lambda,\eta x_i)} - (\beta-1)\eta^{2(\lambda-1)}\sum_{i=1}^n\frac{x_i^{2\lambda}\exp(-2\eta x_i)}{\gamma(\lambda,\eta x_i)^2},$$

$$U_{\lambda\beta} = -n\psi(\lambda) + \sum_{i=1}^n\frac{\gamma'(\lambda,\eta x_i)}{\gamma(\lambda,\eta x_i)},\qquad U_{\eta\beta} = \eta^{\lambda-1}\sum_{i=1}^n\frac{x_i^\lambda\exp(-\eta x_i)}{\gamma(\lambda,\eta x_i)},\qquad U_{\beta\beta} = -\frac{n}{\beta^2},$$

where $\psi''(\cdot)$ is the trigamma function and $\gamma''(\lambda,\eta x_i) = \int_0^{\eta x_i}(\log(t))^2t^{\lambda-1}\exp(-t)\mathrm{d}t$, for $i = 1,\ldots,n$. The asymptotic $100(1-\eta)\%$ confidence intervals for λ, η, β are given by $\widehat{\lambda}\pm z_{\alpha/2}\times[\widehat{\text{var}}(\widehat{\lambda})]^{1/2}$, $\widehat{\eta}\pm z_{\alpha/2}\times\{\widehat{\text{var}}(\widehat{\eta})\}^{1/2}$ and $\widehat{\beta}\pm z_{\alpha/2}\times[\widehat{\text{var}}(\widehat{\beta})]^{1/2}$, where $\text{var}(\cdot)$ is the diagonal element of $\boldsymbol{J}(\widehat{\boldsymbol{\theta}})^{-1}$. If $\eta = 1$, the EGamma distribution reduces to the ESGamma distribution introduced by Nadarajah and Kotz (2006) and the score function and observed information matrix can also be applied for this distribution. They seem to be new results for the ESGamma distribution, since Nadarajah and Kotz (2006) do not consider maximum likelihood estimation in order to estimate the model's parameters.

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