MAT 2457 - GABARITO DA P1 - 2014 - Prova Tipo 03

① - Considere-se $A=\begin{pmatrix}1&0\\0&0\end{pmatrix}$ e $B=\begin{pmatrix}0&0\\0&1\end{pmatrix}$. Logo $A+B=\begin{pmatrix}1&0\\0&1\end{pmatrix}$.

$\det A+\det B=0+0\neq 1=\det(A+B)$, Daí (I) é falsa.

- A seguir, note que se $I=\begin{pmatrix}1&0\\0&1\end{pmatrix}$, $2I=\begin{pmatrix}2&0\\0&2\end{pmatrix}$ e $\therefore \det(2I)=4\neq 2=2\det(I)$.

  Logo, (II) é falsa.

- Seja $A=\begin{pmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{pmatrix}$. Então $\det(A^t)=\begin{vmatrix}a_{11}&a_{21}&a_{31}\\a_{12}&a_{22}&a_{32}\\a_{13}&a_{23}&a_{33}\end{vmatrix}=$

$$=a_{11}\begin{vmatrix}a_{22}&a_{32}\\a_{23}&a_{33}\end{vmatrix}-a_{12}\begin{vmatrix}a_{21}&a_{31}\\a_{23}&a_{33}\end{vmatrix}+a_{13}\begin{vmatrix}a_{21}&a_{31}\\a_{22}&a_{32}\end{vmatrix}=a_{11}\begin{vmatrix}a_{22}&a_{23}\\a_{32}&a_{33}\end{vmatrix}-a_{12}\begin{vmatrix}a_{21}&a_{23}\\a_{31}&a_{33}\end{vmatrix}+$$

$$+a_{13}\begin{vmatrix}a_{21}&a_{22}\\a_{31}&a_{32}\end{vmatrix}=\det(A)$$ ∴ (III) é verdadeira. (Resp. b).

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② $\vec{v}=(2,3,1)$; $\vec{w}=(-2,2,-1)$.

$0=(\vec{u}-\vec{w})\cdot\vec{v}=\vec{u}\cdot\vec{v}-\vec{w}\cdot\vec{v}$ ∴ $\vec{u}\cdot\vec{v}=\vec{w}\cdot\vec{v}=-4+6-1=\boxed{1}$ (Resp. e)

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③ - O sistema $\begin{cases}x-y=0\\x+y=2\\2x=2\end{cases}$ tem 1 única solução (1,1) e é no caso $p=3>2=n$. Logo (I) é falsa.

- O sistema escalonado terá $\underset{\underset{\text{equações}}{\parallel}}{p'}\leq p<n=$ incógnitas. Logo, haverá pelo menos 1 variável livre num sistema que é homogêneo. Segue, pois, que ele terá ∞s soluções. ∴ (II) é verdadeira.

- O sistema $\begin{cases}x+y=1\\2x+2y=2\end{cases}$ tem infinitas soluções e $p=2=n$. $(B\neq 0)$. Logo (III) é falsa.

(Resp. b)

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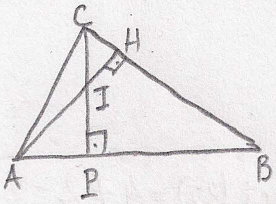
④ $$\det(A)=\begin{vmatrix}x&1&2&x+6\\1&x&2&x+6\\1&2&x&x+6\\1&2&3&x+6\end{vmatrix}=(x+6)\begin{vmatrix}x&1&2&1\\1&x&2&1\\1&2&x&1\\1&2&3&1\end{vmatrix}=(x+6)\begin{vmatrix}x-1&-1&-1&0\\0&x-2&-1&0\\0&0&x-3&0\\1&2&3&1\end{vmatrix}=$$

$$=(x+6)\begin{vmatrix}x-1&-1&-1\\0&x-2&-1\\0&0&x-3\end{vmatrix}=(x+6)(x-1)(x-2)(x-3)$$

(Resp. c)

⑤



$\vec{AB} = (1, -1, 1)$ $\therefore \vec{AC} = (2,1,0)$

$\vec{BC} = (1, 2, -1)$

$\vec{AH} = \left(\frac{4}{3}, \frac{-1}{3}, \frac{2}{3}\right)$

$\vec{AI} = \alpha \cdot \vec{AH} = \left(\frac{4\alpha}{3}, \frac{-\alpha}{3}, \frac{2\alpha}{3}\right).$

Também, $\vec{AI} = \vec{AC} + \vec{CI}$ e $\vec{CI} = \beta \vec{CP}$ e

$\vec{CP} = \vec{CA} + \vec{AP} = -\vec{AC} + \gamma \vec{AB}$. Logo, $\vec{CP} = (-2+\gamma, -1-\gamma, \gamma)$

Agora, $\vec{CP} \perp \vec{AB}$ Logo. $0 = \vec{CP} \cdot \vec{AB} = (\gamma-2) + (\gamma+1) + \gamma = 3\gamma - 1$

Logo, $\gamma = \frac{1}{3}$. Daí, $\vec{CP} = \left(\frac{-5}{3}, \frac{-4}{3}, \frac{1}{3}\right)$.

Segue-se pois que $\vec{CI} = \left(-\frac{5\beta}{3}, -\frac{4\beta}{3}, \frac{\beta}{3}\right)$, de modo que $\vec{AI} = \left(2 - \frac{5\beta}{3}, 1 - \frac{4\beta}{3}, \frac{\beta}{3}\right)$. Assim, $\begin{cases} 4\alpha = 6 - 5\beta \\ -\alpha = 3 - 4\beta \\ 2\alpha = \beta \end{cases}$. Daí, $-\alpha = 3 - 8\alpha$ e $\therefore$ $\boxed{\alpha = \frac{3}{7}}$

Conclusão: $\boxed{\vec{AI} = \left(\frac{4}{7}, \frac{-1}{7}, \frac{2}{7}\right)}$ (Resp b)

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⑥

$$\left(\begin{array}{cccc|cccc} 1 & 1 & 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 2 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{cccc|cccc} 1 & 1 & 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -2 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -2 & -1 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{cccc|cccc} 1 & 1 & 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1/2 & 0 & -1/2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 & -1 & 0 & 0 & 1 \end{array}\right) \sim$$

$$\sim \left(\begin{array}{cccc|cccc} 1 & 1 & 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1/2 & 0 & -1/2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -2 & -1 & -1 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{cccc|cccc} 1 & 1 & 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1/2 & 0 & -1/2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1/2 & 1/2 & 0 & -1/2 \end{array}\right) \sim \left(\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1/2 & 0 & -1/2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1/2 & 1/2 & 0 & -1/2 \end{array}\right) \sim$$

$$\sim \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & \\ 0 & 1 & 0 & 0 & \\ 0 & 0 & 1 & 0 & \\ 0 & 0 & 0 & 1 & \end{array} \underbrace{\begin{array}{cccc} -1/2 & -1 & 1/2 & 1 \\ 1/2 & 0 & -1/2 & 0 \\ 0 & 1 & 0 & 0 \\ 1/2 & 1/2 & 0 & -1/2 \end{array}}_{A^{-1}}\right)$$

$\therefore$ $tr(A^{-1}) = \left(\frac{-1}{2}\right) + 0 + 0 + \left(\frac{-1}{2}\right) = \boxed{-1}$

(Resp. e)

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⑦

- Sejam $\vec{a} = (1,0,0)$ e $\vec{b} = (-1,0,0)$. Entao $\vec{a} + \vec{b} = \vec{0}$. Mas $\vec{a} \not\perp \vec{b}$ $(\vec{a} \cdot \vec{b} = -1 \neq 0)$.
  $\therefore$ (I) é falsa.
- $(\forall \vec{a}, \vec{b})$ $\|\vec{a}\| = \|\vec{a} + \vec{b} + (-\vec{b})\| \leq \|\vec{a} + \vec{b}\| + \|\vec{b}\|$
  Logo, $\|\vec{a}\| - \|\vec{b}\| \leq \|\vec{a} + \vec{b}\| \leq \|\vec{a}\| + \|\vec{b}\|$
  Em nosso caso, $1 = 3 - 2 = \|\vec{a}\| - \|\vec{b}\| \leq \|\vec{a} + \vec{b}\| \leq 3 + 2 = 5$ $\therefore$ (II) é verdad.
- $13 = \|\vec{a} + \vec{b}\|^2 = (\vec{a} + \vec{b}) \cdot (\vec{a} + \vec{b}) = \|\vec{a}\|^2 + 2\vec{a} \cdot \vec{b} + \|\vec{b}\|^2 = 13 + 2\vec{a} \cdot \vec{b} \Leftrightarrow \vec{a} \cdot \vec{b} = 0$
  $\Leftrightarrow \vec{a} \perp \vec{b}$ $\therefore$ (III) é verdad.

(Resp. c)

⑧ (*) $(4\vec{u}+\vec{v}) \perp (2\vec{u}-\frac{1}{2}\vec{v}) \Leftrightarrow 0 = (4\vec{u}+\vec{v})\cdot(2\vec{u}-\frac{1}{2}\vec{v}) = 8\|\vec{u}\|^2 - 2\vec{v}\cdot\vec{u} + 2\vec{u}\cdot\vec{v} - \frac{1}{2}\|\vec{v}\|^2 =$

$= \frac{16\|\vec{u}\|^2 - \|\vec{v}\|^2}{2} \Leftrightarrow 16\|\vec{u}\|^2 = \|\vec{v}\|^2 \Leftrightarrow 4\|\vec{u}\| = \|\vec{v}\|$. $\therefore$ (I) é verd.

(*) $\|\vec{u}+\vec{v}\|^2 + \|\vec{u}-\vec{v}\|^2 = (\vec{u}+\vec{v})\cdot(\vec{u}+\vec{v}) + (\vec{u}-\vec{v})\cdot(\vec{u}-\vec{v}) = \|\vec{u}\|^2 + 2\vec{u}\cdot\vec{v} + \|\vec{v}\|^2 + \|\vec{u}\|^2 - 2\vec{u}\cdot\vec{v} + \|\vec{v}\|^2 =$

$= 2(\|\vec{u}\|^2 + \|\vec{v}\|^2)$ $\therefore$ (II) é verd.

(*) Dados $\vec{u}, \vec{v}$, $\|\vec{u} - 2\vec{v}\|^2 = \|\vec{u}\|^2 - 4\vec{u}\cdot\vec{v} + 4\|\vec{v}\|^2 \Longleftrightarrow \vec{u}\cdot\vec{v} = \|\vec{u}\|\cdot\|\vec{v}\| \Leftrightarrow$

$\|\vec{u}\|^2 - 4\|\vec{u}\|\cdot\|\vec{v}\| + 4\|\vec{v}\|^2$

$\Leftrightarrow \vec{u} = \vec{0}$ ou $\vec{v} = \vec{0}$ ou $(\exists \lambda > 0)\ \vec{u} = \lambda\vec{v}$.

Logo, se tomarmos $\vec{v} \neq \vec{0}$ e $\vec{u} = \lambda\vec{v}$, $\forall \lambda > 0$, teremos INFINITOS pares de $(\vec{u}, \vec{v})$ que satisfazem a igualdade acima.

$\therefore$ (III) é falsa. (Resp. e)

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⑨ $\vec{u} = (2, a, a)$

$\vec{AB} = (1-a, a-1, -a)$

Logo, $\vec{AB} \perp \vec{u} \Leftrightarrow 0 = \vec{u}\cdot\vec{AB} = 2(1-a) + a(1-a) - a^2 \Leftrightarrow$

$\Leftrightarrow 2 - 3a = 0 \Leftrightarrow \boxed{a = \frac{2}{3}}$ (Resp. c)

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⑩ $\vec{AB} = (-2, 0, -6)$

$\vec{AC} = (3, -1, -5)$

$\vec{BD} = (6, -2, 6)$

; $\vec{AB}\cdot\vec{AC} = 24$. Logo, se $\theta = \widehat{\vec{AB}, \vec{AC}}$,

$\cos\theta = \frac{24}{\|\vec{AB}\|\cdot\|\vec{AC}\|} > 0$ $\therefore$ θ é agudo.

Por outro lado, se $(\lambda \in \mathbb{R})$ é t.q. $\vec{BD} = \lambda\vec{AC}$,

então $\begin{cases} 6 = 3\lambda \\ -2 = -\lambda \\ 6 = (-5)\lambda \end{cases}$ $\therefore 2 = \lambda = \frac{-6}{5}$ (absurdo) $\therefore \nexists \lambda$ tq $\vec{BD} = \lambda\vec{AC}$

Logo $\vec{BD}$ e $\vec{AC}$ não são //s.

Por fim, $\vec{AC}\cdot\vec{BD} = 18 + 2 - 30 = -10 \neq 0$ $\therefore \vec{BD} \not\perp \vec{AC}$.

(Resp e)

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⑪ $\det(A) = \begin{vmatrix} 1 & 3 & -2 & -3 \\ 0 & 1 & 0 & -1 \\ 0 & 6 & -1 & 0 \\ 0 & -1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & -1 \\ 6 & -1 & 0 \\ -1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 6 & -1 & 6 \\ -1 & 1 & 0 \end{vmatrix} = -6$.

Notando que $B = A^t$ e lembrando que $\det(A^t) = \det(A)$ e que $\det(A\cdot B) = \det(A)\cdot\det(B)$, segue que $\det(A^2 B) = [\det(A)]^3 = (-6)^3 = -6^3$

$\therefore$ apenas (II) e (III) são verd. (Resp d).

⑫ $\vec{u}$ é comb. linear de $(2,2,4)$ e $(0,1,3)$ $\Leftrightarrow (\exists x,y \in \mathbb{R})\ \vec{u} = x(2,2,4) + y(0,1,3) \Leftrightarrow$

$$\Leftrightarrow (\exists x,y\in R) \begin{cases} 2x = a \\ 2x+y = 1-b \\ 4x+3y = 2c \end{cases} \Leftrightarrow (\exists x,y\in R) \begin{cases} 2x = a \\ 2x+y = 1-b \\ -2x = -3+3b+2c \end{cases} \Leftrightarrow$$

$$\Leftrightarrow (\exists x,y\in \mathbb{R}) \begin{cases} 2x = a \\ 2x+y = 1-b \\ 0 = -3+a+3b+2c \end{cases} \Leftrightarrow \boxed{a+3b+2c = 3}$$

(Resp. d)

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⑬ Denotemos por x, y e z respectivamente o nº de moedas de 1, 5 e 10 cent. (Logo, x, y, z são nºs inteiros positivos).

Daí, $\begin{cases} x+y+z=13 \\ x+5y+10z=83 \end{cases}$. Logo, $\begin{cases} x+y+z=13 \\ 4y+9z=70 \end{cases}$

Assim, $y = \dfrac{70-9z}{4}$ e $x = \dfrac{-18+5z}{4}$

Agora, $1 \leq x = \dfrac{-18+5z}{4}$. De modo que $4 < \dfrac{22}{5} \leq z$. $\therefore z \geq 5$.

e $1 \leq y = \dfrac{70-9z}{4}$. Donde $z \leq \dfrac{66}{9}$ $\therefore z \leq 7$.

Mas se $z=5$ ou $z=7$, x não é inteiro. Segue-se pois que $z=6$ e $\therefore x=3$ e $y=4$. Logo, $x+y-z = 3+4-6 = 1$] (Resp. d)

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⑭ $\vec{u} = (a,b,c)$; $*$ $\vec{u} \perp (1,-1,0) \Leftrightarrow 0 = \vec{u}\cdot(1,-1,0) = a-b$ $\therefore b=a$ $\therefore \vec{u} = (a,a,c)$.

$*$ $a+c = \vec{u}\cdot(1,0,1) = \|\vec{u}\|\cdot\|(1,0,1)\cdot \cos 45° = 4\cdot\sqrt{2}\cdot\dfrac{\sqrt{2}}{2} = 4.$ $\therefore c = 4-a$

$\therefore \vec{u} = (a,a,4-a)$.

$*$ $16 = \|\vec{u}\|^2 = a^2 + a^2 + (4-a)^2 = 3a^2 - 8a + 16$

Logo, $a(3a-8)=0$ $\therefore$ $a=0$ ou $a=\dfrac{8}{3}$ $\therefore$ $\boxed{a+b+c = 4 \text{ ou } \dfrac{20}{3}}$ (Resp. d)

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⑮ $$\det(B) = 5\underbrace{\begin{vmatrix} a & c & b \\ d & f & e \\ g & i & h \end{vmatrix}}_{-\det(A)} - 2\underbrace{\begin{vmatrix} a & c & c \\ d & f & f \\ g & i & i \end{vmatrix}}_{=0} = -10. \therefore \det(B^{-1}) = \frac{1}{\det(B)} = \boxed{\frac{-1}{10}}$$ (Resp e)

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⑯ $$(*) \sim \begin{pmatrix} 1 & 1 & m \\ 1 & 1 & n \\ m & n & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & m \\ 0 & 0 & n-m \\ 0 & n-m & 1-m^2 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & m \\ 0 & n-m & 1-m^2 \\ 0 & 0 & n-m \end{pmatrix}.$$ Assim, se $n \neq m$ $(n-m \neq 0)$ o sist. terá 1 única solução e $\therefore$ (I) é verd.

Agora, se $n=m$, $(*) \sim \begin{pmatrix} 1 & 1 & m \\ 0 & 0 & 1-m^2 \\ 0 & 0 & 0 \end{pmatrix}$. Assim, se $m=-1$, o sist. tem 2 variáveis livres e $\therefore$ (II) é falsa!.

Ainda, se $n=m$ e $m=0$, $(*) \sim \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ e $\therefore$ o sist tem só 1 variável livre (x_2 ou x_1)

e $\therefore$ (III) é falsa.

(Resp. b)