2. **(2,5 pontos)** Seja $f : \mathbb{R}^2 \to \mathbb{R}$, $f = f(x,y)$ uma função de classe C^2 e seja

$$g(t,u) = f(2tu, t^2 - u^2).$$

Calcule as derivadas parciais $\frac{\partial g}{\partial t}, \frac{\partial g}{\partial u}, \frac{\partial^2 g}{\partial t^2}$ e $\frac{\partial^2 g}{\partial u^2}$ em termos das derivadas parciais de f e determine o valor de $r > 0$ para que a igualdade

$$\frac{\partial^2 g}{\partial u^2}(t,u) + \frac{\partial^2 g}{\partial t^2}(t,u) = 56\left[\frac{\partial^2 f}{\partial x^2}(2tu, t^2-u^2) + \frac{\partial^2 f}{\partial y^2}(2tu, t^2-u^2)\right]$$

seja válida para todo $(u,t) \in \mathbb{R}^2$ com $t^2 + u^2 = r^2$.

Observação: Deixe os cálculos de TODAS as derivadas parciais na prova.

$$g(t,u) = f(\underbrace{2tu}_{x}, \underbrace{t^2-u^2}_{y})$$

$$\frac{\partial g}{\partial t}(t,u) = \frac{\partial f}{\partial x}(x,y)\cdot 2u + \frac{\partial f}{\partial y}(x,y)\cdot 2t$$

$$\frac{\partial^2 g}{\partial t^2}(t,u) = 2u\left[\frac{\partial^2 f}{\partial x^2}(x,y)2u + \frac{\partial^2 f}{\partial y\partial x}(x,y)2t\right]$$

$$+ 2\frac{\partial f}{\partial y}(x,y) + 2t\left[\frac{\partial^2 f}{\partial x\partial y}(x,y)2u + \frac{\partial^2 f}{\partial y^2}(x,y)2t\right]$$

Assim:
$$\boxed{\begin{aligned}\frac{\partial^2 g}{\partial t^2}(u,t) &= 4u^2\frac{\partial^2 f}{\partial x^2}(2tu, t^2-u^2) + 4ut\frac{\partial^2 f}{\partial y\partial x}(2tu,t^2-u^2)\\ &+ 4ut\frac{\partial^2 f}{\partial x\partial y}(2tu,t^2-u^2) + 4t^2\frac{\partial^2 f}{\partial y^2}(2tu,t^2-u^2)\\ &+ 2\frac{\partial f}{\partial y}(x,y).\end{aligned}} \quad (1)$$

$$\frac{\partial g}{\partial u}(t,u) = \frac{\partial f}{\partial x}(x,y)\cdot 2t + \frac{\partial f}{\partial y}(x,y)(-2u)$$

$$\frac{\partial^2 g}{\partial u^2}(t,u) = 2t\left[\frac{\partial^2 f}{\partial x^2}(x,y)2t + \frac{\partial^2 f}{\partial y\partial x}(x,y)(-2u)\right] +$$

$$-2\frac{\partial f}{\partial y}(x,y) - 2u\left[\frac{\partial^2 f}{\partial x \partial y}(x,y)\,2t + \frac{\partial^2 f}{\partial y^2}(x,y)(-2u)\right]$$

Logo
$$\boxed{\begin{aligned}\frac{\partial^2 g}{\partial u^2}(t,u) &= 4t^2\frac{\partial^2 f}{\partial x^2}(2tu, t^2-u^2) - 4tu\frac{\partial^2 f}{\partial y \partial x}(2tu,t^2-u^2)\\ &-4tu\frac{\partial^2 f}{\partial x \partial y}(2tu,t^2-u^2) + 4u^2\frac{\partial^2 f}{\partial y^2}(2tu,t^2-u^2) - 2\frac{\partial f}{\partial y}(2tu,t^2-u^2)\end{aligned}}$$

(2)

É claro que (1) + (2) $\Rightarrow$

$$\frac{\partial^2 g}{\partial t^2}(t,u) + \frac{\partial^2 g}{\partial u^2}(t,u) = 4(t^2+u^2)\frac{\partial^2 f}{\partial x^2}(2tu,t^2-u^2)$$
$$+ 4(t^2+u^2)\frac{\partial^2 f}{\partial y^2}(2tu,t^2-u^2)$$

$\Rightarrow$

$$\boxed{\frac{\partial^2 g}{\partial t^2}(t,u) + \frac{\partial^2 g}{\partial u^2}(t,u) = 4(t^2+u^2)\left[\frac{\partial^2 f}{\partial x^2}(2tu,t^2-u^2) + \frac{\partial^2 f}{\partial y^2}(2tu,t^2-u^2)\right]}$$

Se $t^2+u^2 = r^2$, queremos então que
$4r^2 = 56$, logo $r = \sqrt{14}$.