2. **(2,5 pontos)** Seja $f : \mathbb{R}^2 \to \mathbb{R}$, $f = f(x,y)$ uma função de classe $C^2$ e seja

$$g(u,t) = f(u^2 - t^2, 2ut).$$

Calcule as derivadas parciais $\frac{\partial g}{\partial u}, \frac{\partial g}{\partial t}, \frac{\partial^2 g}{\partial u^2}$ e $\frac{\partial^2 g}{\partial t^2}$ em termos das derivadas parciais de $f$ e determine o valor de $r > 0$ para que a igualdade

$$\frac{\partial^2 g}{\partial u^2}(u,t) + \frac{\partial^2 g}{\partial t^2}(u,t) = 48\left[\frac{\partial^2 f}{\partial x^2}(u^2 - t^2, 2ut) + \frac{\partial^2 f}{\partial y^2}(u^2 - t^2, 2ut)\right]$$

seja válida para todo $(u,t) \in \mathbb{R}^2$ com $u^2 + t^2 = r^2$.

**Observação:** Deixe os cálculos de TODAS as derivadas parciais na prova.

$$g(u,t) = f(\underbrace{u^2-t^2}_{x}, \underbrace{2ut}_{y})$$

$$\frac{\partial g}{\partial u}(u,t) = \frac{\partial f}{\partial x}(x,y)\,2u + \frac{\partial f}{\partial y}(x,y)\,2t$$

$$\frac{\partial^2 g}{\partial u^2}(u,t) = 2\frac{\partial f}{\partial x}(x,y) + 2u\left[\frac{\partial^2 f}{\partial x^2}(x,y)\,2u + \frac{\partial^2 f}{\partial y\partial x}(x,y)\,2t\right] + 2t\left[\frac{\partial^2 f}{\partial x\partial y}(x,y)\,2u + \frac{\partial^2 f}{\partial y^2}(x,y)\cdot 2t\right]$$

Logo $$\boxed{\frac{\partial^2 g}{\partial u^2}(u,t) = 2\frac{\partial f}{\partial x}(u^2-t^2, 2ut) + 4u^2\frac{\partial^2 f}{\partial x^2}(u^2-t^2, 2ut) + 4ut\frac{\partial^2 f}{\partial y\partial x}(u^2-t^2, 2ut) + 4t^2\frac{\partial^2 f}{\partial y^2}(u^2-t^2, 2ut).}$$ (1)

$$\frac{\partial g}{\partial t}(u,t) = \frac{\partial f}{\partial x}(x,y)(-2t) + \frac{\partial f}{\partial y}(x,y)\,2u$$

$$\frac{\partial^2 g}{\partial t^2}(u,t) = -2\frac{\partial f}{\partial x}(x,y) - 2t\left[\frac{\partial^2 f}{\partial x^2}(x,y)(-2t) + \frac{\partial^2 f}{\partial y\partial x}(x,y)\,2u\right] + 2u\left[\frac{\partial^2 f}{\partial x\partial y}(x,y)\cdot(-2t) + \frac{\partial^2 f}{\partial y^2}(x,y)\,2u\right]$$

Assim: $$\boxed{\frac{\partial^2 g}{\partial t^2}(u,t) = -2\frac{\partial f}{\partial x}(u^2-t^2, 2ut) + 4t^2\frac{\partial^2 f}{\partial x^2}(u^2-t^2, 2ut) - 4ut\frac{\partial^2 f}{\partial x\partial y}(u^2-t^2, 2ut) + 4u^2\frac{\partial^2 f}{\partial y^2}(u^2-t^2, 2ut).}$$ (2)

Fazendo (1) + (2) obtemos:

$$\frac{\partial^2 g}{\partial u^2}(u,t) + \frac{\partial^2 g}{\partial t^2}(u,t) = 4(u^2+t^2)\frac{\partial^2 f}{\partial x^2}(u^2-t^2, 2ut)$$

$$+4(u^2+t^2)\frac{\partial^2 f}{\partial y^2}(u^2-t^2, 2ut)$$

$$= 4(u^2+t^2)\left[\frac{\partial^2 f}{\partial x^2}(u^2-t^2, 2ut) + \frac{\partial^2 f}{\partial y^2}(u^2-t^2, 2ut)\right]$$

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Queremos então $4(u^2+t^2) = 48$ se

$u^2+t^2 = r^2$. Logo $4r^2 = 48$ e

então $r = \sqrt{12}$.