1. **(3,0)** Seja $f(x,y) = ye^{\sqrt[3]{x^4+y^2}}$.

   (a) Determine $\frac{\partial f}{\partial y}$ explicitando o seu domínio. A função $\frac{\partial f}{\partial y}$ é contínua em $(0,0)$?

   (b) Determine o conjunto dos pontos de $\mathbb{R}^2$ nos quais f é diferenciável. **Justifique !**

(a) Se $(x,y) \neq (0,0)$

$$\frac{\partial f}{\partial y}(x,y) = e^{\sqrt[3]{x^4+y^2}} + y e^{\sqrt[3]{x^4+y^2}} \cdot \frac{1}{3}(x^4+y^2)^{-2/3} \cdot 2y$$

$$= e^{\sqrt[3]{x^4+y^2}}\left(1 + \frac{2}{3}\frac{y^2}{(x^4+y^2)^{2/3}}\right)$$

Em $(0,0)$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{y\to 0} \frac{f(0,y)-f(0,0)}{y-0} = \lim_{y\to 0} \frac{y e^{\sqrt[3]{x^4+y^2}}}{y} = 1$$

Assim, o domínio de $\frac{\partial f}{\partial y}$ é $\mathbb{R}^2$ e

$$\frac{\partial f}{\partial y}(x,y) = \begin{cases} e^{\sqrt[3]{x^4+y^2}}\left(1 + \frac{2}{3}\frac{y^2}{(x^4+y^2)^{2/3}}\right) & \text{se } (x,y)\neq(0,0) \\ 1 & \text{se } (x,y) = (0,0). \end{cases}$$

Verificar se $\frac{\partial f}{\partial y}$ é contínua em $(0,0)$.

Ver se $\lim_{(x,y)\to(0,0)} \frac{\partial f}{\partial y}(x,y) = \frac{\partial f}{\partial y}(0,0)$.

$$\lim_{(x,y)\to(0,0)} e^{\overset{\to 0}{\sqrt[3]{x^4+y^2}}}\Big(1 + \frac{2}{3}\underbrace{\frac{y^2}{(x^4+y^2)^{2/3}}}_{(*)}\overset{\to 0}{}\Big) = 1 = \frac{\partial f}{\partial y}(0,0)$$

(*) $$\lim_{(x,y)\to(0,0)} \frac{y^2}{(x^4+y^2)^{2/3}} = \lim_{(x,y)\to(0,0)} \sqrt[3]{\overset{\to 0}{y^2}\underbrace{\left(\frac{y^2}{x^4+y^2}\right)^2}_{\text{limitada } (**)}} = 0$$

(**) $0 \leq y^2 \leq x^4+y^2$. Então $0 \leq \frac{y^2}{x^4+y^2} \leq 1 \quad \forall (x,y)\neq(0,0)$

$$\Rightarrow 0 \leq \left(\frac{y^2}{x^4+y^2}\right)^2 \leq 1$$

(b) $\frac{\partial f}{\partial y}$ é contínua em todos os pontos de $\mathbb{R}^2$.

$$\frac{\partial f}{\partial x}(x,y) = y\, e^{\sqrt[3]{x^4+y^2}} \cdot \frac{1}{3}(x^4+y^2)^{-2/3} \cdot 4x^3$$

$$= \frac{4}{3}\frac{x^3 y}{(x^4+y^2)^{2/3}} e^{\sqrt[3]{x^4+y^2}} \quad \text{se } (x,y) \neq (0,0)$$

$$\frac{\partial f}{\partial x}(0,0) = \lim_{x\to 0} \frac{f(x,0)-f(0,0)}{x-0} = \lim_{x\to 0} 0 = 0$$

Assim $$\frac{\partial f}{\partial x}(x,y) = \begin{cases} \frac{4}{3}\frac{x^3 y}{(x^4+y^2)^{2/3}} e^{\sqrt[3]{x^4+y^2}} & \text{se } (x,y)\neq(0,0) \\ 0 & \text{se } (x,y)=(0,0) \end{cases}$$

Observe que (*)

$$\lim_{(x,y)\to(0,0)} \frac{x^3 y}{(x^4+y^2)^{2/3}} = \lim_{(x,y)\to(0,0)} \sqrt[3]{\frac{x^9 y^3}{(x^4+y^2)^2}}$$

$$= \lim_{(x,y)\to(0,0)} \sqrt[3]{\underbrace{\left(\frac{x^4}{(x^4+y^2)}\right)^2}_{\text{limitada}} \underbrace{x y^3}_{\to 0}} = 0$$

Assim $$\lim_{(x,y)\to(0,0)} \frac{\partial f}{\partial x}(x,y) = \lim_{(x,y)\to(0,0)} \frac{4}{3}\underbrace{\frac{x^3 y}{(x^4+y^2)^{2/3}}}_{\to 0 \text{ (por (*))}} \underbrace{e^{\sqrt[3]{x^4+y^2}}}_{\to 1}$$

$$= 0 \cdot 1 = 0 = \frac{\partial f}{\partial x}(0,0).$$

As funções $\frac{\partial f}{\partial x}$ e $\frac{\partial f}{\partial y}$ são contínuas em todos os pontos de $\mathbb{R}^2$. Logo f é diferenciável em $\mathbb{R}^2$.

(outra solução na prova A)