-A- e -B-

Questão 1.

a) (1,5) Seja $p(x)$ o polinômio de Taylor de ordem 4 da função $f(x) = \sqrt{x}$ em torno do ponto $x_0 = 1$. Determine $p(x)$ e mostre que:

$$|f(x) - p(x)| < \frac{7}{2^8}(x-1)^5, \text{ para todo } x > 1.$$

b) (0,5) Use o item (a), com $x = 1 + t^4$, para encontrar um valor aproximado de $\int_0^{\frac{1}{2}} \sqrt{1+t^4}\,dt$.

c) (0,5) Verifique que o erro na estimativa feita em (b) é inferior a $\frac{1}{2^{30}}$.

$f(x) = \sqrt{x}$, $x_0 = 1$ $\Rightarrow$ $f(x_0) = 1$

$f'(x) = \frac{1}{2\sqrt{x}}$ $\Rightarrow$ $f'(x_0) = \frac{1}{2}$

$f''(x) = -\frac{1}{4} x^{-3/2}$ $\Rightarrow$ $f''(x_0) = -\frac{1}{4}$

$f'''(x) = \frac{3}{8} x^{-5/2}$ $\Rightarrow$ $f'''(x_0) = \frac{3}{8}$

$f^{(4)}(x) = -\frac{15}{16} x^{-7/2}$ $\Rightarrow$ $f^{(4)}(x_0) = -\frac{15}{16}$

$f^{(5)}(x) = \frac{15}{16} \cdot \frac{7}{2} x^{-9/2} = \frac{15 \cdot 7}{2^5} x^{-\frac{9}{2}}$.

Portanto:

(a) $\boxed{p(x) = 1 + \frac{1}{2}(x-1) - \frac{1}{4}\frac{(x-1)^2}{2} + \frac{3}{8}\frac{(x-1)^3}{3!} - \frac{15}{16}\frac{(x-1)^4}{4!}}$

Sabemos que $f(x) - p(x) = f^{(5)}(\bar{x}) \frac{(x-1)^5}{5!}$, para algum $\bar{x}$ entre 1 e x.

Se $x > 1$, então $1 < \bar{x} < x$

$\Rightarrow 1 < \sqrt{\bar{x}} < \sqrt{x}$

$\Rightarrow 1 < (\sqrt{\bar{x}})^9 < (\sqrt{x})^9$

$\Rightarrow (0<) \frac{1}{(\sqrt{\bar{x}})^9} < 1$

Logo,

$|f(x) - p(x)| = \frac{15 \cdot 7}{2^5} (\bar{x})^{-9/2} \frac{(x-1)^5}{5!} < \frac{\cancel{15} \cdot 7}{2^5} \cdot \frac{(x-1)^5}{\cancel{5} \cdot 4 \cdot \cancel{3} \cdot 2}$

$\Rightarrow \boxed{|f(x) - p(x)| < \frac{7}{2^8}(x-1)^5 \text{ , se } x > 1.}$

Questão 1 - continuação

Fazendo $x = 1 + t^4$ $\quad (\Rightarrow x - 1 = t^4)$ temos:

$$|f(1+t^4) - p(1+t^4)| < \frac{7}{2^8}(t^4)^5 = \frac{7}{2^8}t^{20}$$

$$\Leftrightarrow -\frac{7}{2^8}t^{20} < \sqrt{1+t^4} - p(1+t^4) < \frac{7}{2^8}t^{20}.$$

Calculando-se a integral, iremos obter:

$$\underbrace{-\int_0^{\frac{1}{2}} \frac{7}{2^8}t^{20}\,dt}_{(*)} \leq \int_0^{1/2} \sqrt{1+t^4}\,dt - \int_0^{\frac{1}{2}} p(1+t^4)\,dt \leq \underbrace{\int_0^{\frac{1}{2}} \frac{7}{2^8}t^{20}\,dt}_{(*)},$$

$$(*) \int_0^{\frac{1}{2}} \frac{7}{2^8}t^{20}\,dt = \frac{\cancel{7}}{2^8}\frac{t^{21}}{\cancel{21}}\Big|_0^{\frac{1}{2}} = \frac{1}{2^8}\cdot\frac{1}{3}\cdot\left(\frac{1}{2}\right)^{21} < \frac{1}{2^{30}}$$

Portanto,

$$-\frac{1}{2^{30}} < \int_0^{1/2} \sqrt{1+t^4}\,dt - \int_0^{1/2} p(1+t^4)\,dt < \frac{1}{2^{30}},$$

ou seja, $\boxed{\left| \int_0^{1/2} \sqrt{1+t^4}\,dt - \int_0^{1/2} p(1+t^4)\,dt \right| < \frac{1}{2^{30}}}$ $\quad$ (c)

e um valor aproximado para $\int_0^{1/2} \sqrt{1+t^4}\,dt$ é

$$\int_0^{\frac{1}{2}} p(1+t^4)\,dt = \int_0^{\frac{1}{2}} \left[1 + \frac{1}{2}t^4 - \frac{1}{4}\frac{(t^4)^2}{2} + \frac{3}{8}\frac{(t^4)^3}{3!} - \frac{15}{16}\frac{(t^4)^4}{4!}\right] dt =$$

$$= \int_0^{1/2} \left[1 + \frac{t^4}{2} - \frac{1}{8}t^8 + \frac{\cancel{3}}{8}\cdot\frac{1}{\cancel{3!}}t^{12} - \frac{15}{16}\cdot\frac{1}{4!}t^{16}\right] dt =$$

$$= \left[t + \frac{t^5}{10} - \frac{t^9}{9\cdot 8} + \frac{1}{16}\frac{t^{13}}{13} - \frac{15}{16}\cdot\frac{1}{4!}\frac{t^{17}}{17}\right]_0^{1/2} =$$

$$\boxed{= \frac{1}{2} + \frac{1}{10}\cdot\frac{1}{2^5} - \frac{1}{72}\cdot\frac{1}{2^9} + \frac{1}{13}\cdot\frac{1}{2^4}\cdot\frac{1}{2^{13}} - \frac{15}{2^4}\cdot\frac{1}{4!}\cdot\frac{1}{17}\cdot\frac{1}{2^{17}}}$$

(b)