4. **(2,5)** Sejam $f : \mathbb{R}^2 \to \mathbb{R}, f = f(x,y)$, uma função de classe C^2 e $F : \mathbb{R}^2 \to \mathbb{R}$ dada por

$$F(t,u) = u^2 f(u^2 t, 2t + 4u).$$

Calcule $\dfrac{\partial^2 F}{\partial t \partial u}(1,2)$ em termos de f e de suas derivadas parciais $\dfrac{\partial f}{\partial x}$ e $\dfrac{\partial f}{\partial y}$.

Como $f \in C^2$, f e suas derivadas parciais de 1° ordem são deriváveis. Aplicando-se a regra de Leibnitz e a regra da cadeia, conclui-se que, $\forall (t,u) \in \mathbb{R}^2$:

1.) $$\frac{\partial F}{\partial u}(t,u) = 2u \cdot f(u^2t, 2t+4u) + u^2 \cdot \left[\frac{\partial f}{\partial x}(u^2t, 2t+4u) \cdot 2ut + \frac{\partial f}{\partial y}(u^2t, 2t+4u) \cdot 4 \right]$$

2.) $$\frac{\partial^2 F}{\partial t \partial u}(t,u) = 2u \cdot \left[\frac{\partial f}{\partial x}(u^2t, 2t+4u) \cdot u^2 + \frac{\partial f}{\partial y}(u^2t, 2t+4u) \cdot 2 \right] + 2u^3 \frac{\partial f}{\partial x}(u^2t, 2t+4u) + 2u^3 t \left[\frac{\partial^2 f}{\partial x^2}(u^2t, 2t+4u) \cdot u^2 + \frac{\partial^2 f}{\partial y \partial x}(u^2t, 2t+4u) \cdot 2 \right] + 4u^2 \cdot \left[\frac{\partial^2 f}{\partial x \partial y}(u^2t, 2t+4u) \cdot u^2 + \frac{\partial^2 f}{\partial y^2}(u^2t, 2t+4u) \cdot 2 \right]$$

$$\therefore \frac{\partial^2 F}{\partial t \partial u}(1,2) = 32 \frac{\partial f}{\partial x}(4,10) + 8 \frac{\partial f}{\partial y}(4,10) + 64 \frac{\partial^2 f}{\partial x^2}(4,10) + 96 \frac{\partial^2 f}{\partial x \partial y}(4,10) + 32 \frac{\partial^2 f}{\partial y^2}(4,10)$$