Questão 3. Calcule, caso exista, ou mostre que não existe

a) (0,5 ponto) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\operatorname{sen}(y^4 - x^4)}{y^2 - x^2}$

b) (1 ponto) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{y^2 - x^2 y}{\sqrt{x^4 + y^2}}$

c) (1 ponto) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y^2 - 2x^3y}{x^6 + y^2}$

a) $$\lim_{(x,y)\to(0,0)} \frac{\operatorname{sen}(y^4-x^4)}{y^2-x^2} = \lim_{(x,y)\to(0,0)} (y^2+x^2)\,\frac{\operatorname{sen}(y^4-x^4)}{(y^4-x^4)}$$

Como $\displaystyle\lim_{(x,y)\to(0,0)} y^4-x^4 = 0$, temos $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\operatorname{sen}(y^4-x^4)}{y^4-x^4} = \lim_{u\to 0} \frac{\operatorname{sen} u}{u} = 1$

Logo, $\displaystyle\lim_{(x,y)\to(0,0)} (x^2+y^2)\,\frac{\operatorname{sen}(y^4-x^4)}{y^4-x^4} = 0 \cdot 1 = 0$

b) $$\lim_{(x,y)\to(0,0)} \frac{y^2-x^2y}{\sqrt{x^4+y^2}} = \lim_{(x,y)\to(0,0)} \frac{y}{\sqrt{x^4+y^2}}\,(y-x^2)$$

Temos $0 \leq y^2 \leq x^4+y^2 \underset{(x,y)\neq(0,0)}{\Longrightarrow} 0 \leq \dfrac{y^2}{x^4+y^2} \leq 1 \Rightarrow \left|\dfrac{y}{\sqrt{x^4+y^2}}\right| \leq 1$

e $\displaystyle\lim_{(x,y)\to(0,0)} y^2 - x = 0$

Logo, $\displaystyle\lim_{(x,y)\to(0,0)} \frac{y}{\sqrt{x^4+y^2}}\,\overset{\to 0}{(y^2-x^2)} = 0$

c) Sejam $\gamma_1(t) = (t,0)$, $\gamma_2(t) = (t,t^3)$ e $f(x,y) = \dfrac{x^2y^2 - 2x^3y}{x^6+y^2}$

Temos: $\gamma_1(0) = (0,0)$, $\gamma_2(0) = (0,0)$, γ_1 e γ_2 são contínuas, $\gamma_1(t) \neq (0,0)$ e $\gamma_2(t) \neq (0,0)$ se $t \neq 0$ e $\gamma_1(t), \gamma_2(t) \in \operatorname{Dom} f$, $\forall t \neq 0$.

Como $\displaystyle\lim_{t\to 0} f(\gamma_1(t)) = \lim_{t\to 0} 0 = 0$

e $\displaystyle\lim_{t\to 0} f(\gamma_2(t)) = \lim_{t\to 0} \frac{t^8 - 2t^6}{2t^6} = \lim_{t\to 0} \frac{t^2-2}{2} = -1,$

não existe $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y^2 - 2x^3y}{x^6+y^2}$