Questão 1. Seja $f(x) = (1-x)^{-\frac{1}{2}}$.

a) (1,5 ponto) Seja $p(x)$ o polinômio de Taylor de ordem 5 de f em torno de $x_0 = 0$. Encontre $p(x)$ e mostre que se $0 \le x \le \frac{1}{4}$ então

$$|(1-x)^{-\frac{1}{2}} - p(x)| \le \frac{2^3 \cdot 7 \cdot 11}{3^{\frac{11}{2}}} x^6$$

b) (1 ponto) Mostre que

$$\int_0^{\frac{1}{2}} \frac{1}{\sqrt{1-x^2}}\, dx \cong \frac{1}{2} + \frac{1}{2^4 \cdot 3} + \frac{3}{2^8 \cdot 5} + \frac{5}{2^{11} \cdot 7} + \frac{5 \cdot 7}{2^{16} \cdot 9} + \frac{7 \cdot 9}{2^{19} \cdot 11}$$

com erro, em módulo, inferior a $\frac{7 \cdot 11}{2^{10} \cdot 3^{\frac{11}{2}} \cdot 13}$.

(Curiosidade: Você acaba de encontrar um valor aproximado para $\frac{\pi}{6}$, com erro inferior a 10^{-4}.)

a) $f'(x) = -\frac{1}{2}(1-x)^{-3/2} \cdot (-1) = \frac{1}{2}(1-x)^{-3/2}$; $f''(x) = \frac{3}{2^2}(1-x)^{-5/2}$;

$f'''(x) = \frac{3.5}{2^3}(1-x)^{-7/2}$; $f^{(4)}(x) = \frac{3.5.7}{2^4}(1-x)^{-9/2}$;

$f^{(5)}(x) = \frac{3.5.7.9}{2^5}(1-x)^{-11/2}$; $f^{(6)}(x) = \frac{3.5.7.9.11}{2^6}(1-x)^{-13/2}$

$p(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + f'''(0)\frac{x^3}{3!} + f^{(4)}(0)\frac{x^4}{4!} + f^{(5)}(0)\frac{x^5}{5!}$

$p(x) = 1 + \frac{1}{2}x + \frac{3}{2^2}\frac{1}{2!}x^2 + \frac{3.5}{2^3}\frac{x^3}{3!} + \frac{3.5.7}{2^4}\frac{x^4}{4!} + \frac{3.5.7.9}{2^5}\frac{x^5}{5!}$

Se $0 < x \le \frac{1}{4}$, pela Fórmula de Taylor, temos

$f(x) - p(x) = \frac{3.5.7.9.11}{2^6}\frac{(1-c)^{-13/2}}{6!}$ para algum c, com $0 < c < x \le \frac{1}{4}$

Mas, se $0 < c < \frac{1}{4}$ então $(1-c) > \frac{3}{4}$ e portanto

$0 < (1-c)^{-13/2} < \left(\frac{3}{4}\right)^{-13/2}$. Logo, se $0 < x \le \frac{1}{4}$, temos

$0 \le f(x) - p(x) \le \frac{3.5.7.9.11}{2^6.6!}\left(\frac{3}{4}\right)^{-13/2} x^6 = \frac{3.5.7.9.11}{2^6.6.5.4.3.2}\frac{2^{13}}{3^{13/2}}x^6$

$= \frac{7.11.3^2}{2^{10}.3}\frac{2^{13}}{3^{13/2}}x^6 = \frac{2^3.7.11}{3^{11/2}}x^6$

(Se $x = 0$, temos $(1-x)^{-1/2} - p(x) = 0$)

b) Se $0 \le x \le \frac{1}{2}$ então $0 \le x^2 \le \frac{1}{4}$ e portanto

$$0 \le f(x^2) - p(x^2) \le \frac{2^3 \cdot 7 \cdot 11}{3^{11/2}} x^{12}.$$ Logo

$$0 \le \int_0^{1/2} [f(x^2) - p(x^2)] dx \le \int_0^{1/2} \frac{2^3 \cdot 7 \cdot 11}{3^{11/2}} x^{12} dx = \frac{2^3 \cdot 7 \cdot 11}{3^{11/2} \cdot 13} \frac{1}{2^{13}} =$$

$$= \frac{7 \cdot 11}{2^{10} \, 3^{11/2} \cdot 13}$$

Temos, então $\int_0^{1/2} f(x^2) dx \approx \int_0^{1/2} p(x^2) dx$, com erro, em módulo inferior a $\frac{7 \cdot 11}{2^{10} \, 3^{11/2} \, 13}$, onde $\int_0^{1/2} f(x^2) dx = \int_0^{1/2} \frac{1}{\sqrt{1-x^2}} dx$

$$\int_0^{1/2} p(x^2) dx = \int_0^{1/2} \left[1 + \frac{x^2}{2} + \frac{3 \cdot x^4}{2^2 \cdot 2!} + \frac{3 \cdot 5 x^6}{2^3 3!} + \frac{3 \cdot 5 \cdot 7 x^8}{2^4 \cdot 4!} + \frac{3 \cdot 5 \cdot 7 \cdot 9 \, x^{10}}{2^5 \cdot 5!}\right] dx$$

$$\frac{1}{2} + \frac{1}{2 \cdot 3} \frac{1}{2^3} + \frac{3}{2^2 \cdot 2! \, 5} \frac{1}{2^5} + \frac{3 \cdot 5}{2^3 3! \, 7} \frac{1}{2^7} + \frac{3 \cdot 5 \cdot 7}{2^4 \, 4! \, 9} \frac{1}{2^9} + \frac{3 \cdot 5 \cdot 7 \cdot 9}{2^5 5! \, 11}$$

$$\frac{1}{2} + \frac{1}{2^4 \cdot 3} + \frac{3}{2^8 \cdot 5} + \frac{\cancel{3} \cdot 5}{2^{11} \cdot \cancel{3} \cdot 7} + \frac{5 \cdot 7}{2^{16} \cdot 9} + \frac{7 \cdot 9}{2^{19} \cdot 11}$$