4. **(2,0)** Calcule ou mostre que não existe:

(a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^3+y^4+x^4}{x^3y-xy^3}$

(b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{y^3+\operatorname{sen}(x^2+y^2)}{x^4+\operatorname{sen}(x^2+y^2)}$

(a) Sejam $f(x,y)=\dfrac{xy^3+y^4+x^4}{x^3y-xy^3}$, $\gamma_1(t)=(t,2t)$ e $\gamma_2(t)=(t,3t)$. Temos que: $\gamma_1(0)=(0,0)=\gamma_2(0)$; γ_1 e γ_2 são contínuas; $\gamma_1(t)\neq(0,0)$ e $\gamma_2(t)\neq(0,0)$ se $t\neq 0$ e, para $t\neq 0$, $\gamma_1(t)$ e $\gamma_2(t)$ estão no domínio de f pois, para $t\neq 0$, $t^3 2t - t(2t)^3 = -6t^4 \neq 0$ e $t^3 . 3t - t(3t)^3 = -24t^4 \neq 0$. Como

$$\lim_{t\to 0} f(\gamma_1(t)) = \lim_{t\to 0} \frac{t\,2^3t^3+2^4t^4+t^4}{t^3 2t - t\,2^3t^3} = \frac{-25}{6},$$

$$\lim_{t\to 0} f(\gamma_2(t)) = \lim_{t\to 0} \frac{t\,3^3t^3+3^4t^4+t^4}{t^3 3t - t\,3^3t^3} = \frac{-109}{24}, \text{ e}$$

$\dfrac{-25}{6} = \dfrac{-100}{24} \neq \dfrac{-109}{24}$ então não existe $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y)$.

(b) i) $$\frac{y^3+\operatorname{sen}(x^2+y^2)}{x^4+\operatorname{sen}(x^2+y^2)} = \frac{\dfrac{y^3}{x^2+y^2}+\dfrac{\operatorname{sen}(x^2+y^2)}{x^2+y^2}}{\dfrac{x^4}{x^2+y^2}+\dfrac{\operatorname{sen}(x^2+y^2)}{x^2+y^2}} \quad \text{se } (x,y)\neq(0,0)$$

ii) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{y^2}{x^2+y^2}\cdot y = 0$ pois $\displaystyle\lim_{(x,y)\to(0,0)} y = 0$ e $\left|\dfrac{y^2}{x^2+y^2}\right| \leq 1$

iii) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2}\cdot x^2 = 0$ pois $\displaystyle\lim_{(x,y)\to(0,0)} x^2 = 0$ e $\left|\dfrac{x^2}{x^2+y^2}\right| \leq 1$

iii) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\operatorname{sen}(x^2+y^2)}{x^2+y^2} = 1$ pois $\displaystyle\lim_{t\to 0} \frac{\operatorname{sen} t}{t} = 1$,

$\displaystyle\lim_{(x,y)\to(0,0)} (x^2+y^2) = 0$ e $x^2+y^2 \neq 0$ se $(x,y) \neq (0,0)$

Logo por i) ii) iii)

$$\lim_{(x,y)\to(0,0)} \frac{y^3+\operatorname{sen}(x^2+y^2)}{x^4+\operatorname{sen}(x^2+y^2)} = \lim_{(x,y)\to(0,0)} \frac{\dfrac{y^3}{x^2+y^2} + \dfrac{\operatorname{sen}(x^2+y^2)}{x^2+y^2}}{\dfrac{x^4}{x^2+y^2} + \dfrac{\operatorname{sen}(x^2+y^2)}{x^2+y^2}}$$

$$= \frac{0+1}{0+1} = \underline{1}$$