4. **(2,0)** Calcule ou mostre que não existe:

(a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3y+y^4+x^4}{x^3y-xy^3}$

(b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3+\operatorname{sen}(x^2+y^2)}{y^4+\operatorname{sen}(x^2+y^2)}$

(a) Sejam $f(x,y)=\dfrac{x^3y+y^4+x^4}{x^3y-xy^3}$, $\gamma_1(t)=(t,2t)$ e $\gamma_2(t)=(t,3t)$. Temos que: $\gamma_1(0)=(0,0)=\gamma_2(0)$; γ_1 e γ_2 são contínuas; $\gamma_1(t)\neq(0,0)$ e $\gamma_2(t)\neq(0,0)$ se $t\neq 0$ e, para $t\neq 0$, $\gamma_1(t)$ e $\gamma_2(t)$ estão no domínio de f pois, para $t\neq 0$, $t^3.2t-t.(2t)^3=-6t^4\neq 0$ e $t^3.3t-t(3t)^3=-24t^4\neq 0$. Como

$$\lim_{t\to0} f(\gamma_1(t))=\lim_{t\to0}\frac{t^3.2t+2^4t^4+t^4}{t^3 2t-t\,2^3t}=\lim_{t\to0}\frac{-19}{6},$$

$$\lim_{t\to0} f(\gamma_2(t))=\lim_{t\to0}\frac{t^3 3t+3^4t^4+t^4}{t^3 3t-t\,3^3t^3}=\lim_{t\to0}\frac{-85}{24} \quad \text{e}$$

$\dfrac{-19}{6}=\dfrac{-76}{24}\neq\dfrac{-85}{24}$ então não existe

$\displaystyle\lim_{(x,y)\to(0,0)} f(x,y)$.

(b) i) $$\frac{x^3+\operatorname{sen}(x^2+y^2)}{y^4+\operatorname{sen}(x^2+y^2)}=\frac{\dfrac{x^3}{x^2+y^2}+\dfrac{\operatorname{sen}(x^2+y^2)}{x^2+y^2}}{\dfrac{y^4}{x^2+y^2}+\dfrac{\operatorname{sen}(x^2+y^2)}{x^2+y^2}} \quad \text{se } (x,y)\neq(0,0)$$

ii) $\displaystyle\lim_{(x,y)\to(0,0)}\frac{x^2}{x^2+y^2}x=0$ pois $\displaystyle\lim_{(x,y)\to(0,0)} x=0$ e $\left|\dfrac{x^2}{x^2+y^2}\right|\leq 1$

iii) $\displaystyle\lim_{(x,y)\to(0,0)}\frac{y^2}{x^2+y^2}=0$ pois $\displaystyle\lim_{(x,y)\to(0,0)} y^2=0$ e $\left|\dfrac{y^2}{x^2+y^2}\right|\leq 1$

iii) $\lim\limits_{(x,y)\to(0,0)} \dfrac{\operatorname{sen}(x^2+y^2)}{x^2+y^2} = 1$, pois $\lim\limits_{t\to 0} \dfrac{\operatorname{sen}(t)}{t} = 1$,

$\lim\limits_{(x,y)\to(0,0)} (x^2+y^2) = 0$ e $x^2+y^2 \neq 0$ se $(x,y)\neq(0,0)$

Logo, por i) ii) iii).

$$\lim_{(x,y)\to(0,0)} \frac{x^3 + \operatorname{sen}(x^2+y^2)}{y^4 + \operatorname{sen}(x^2+y^2)} = \lim_{(x,y)\to(0,0)} \frac{\dfrac{x^3}{x^2+y^2} + \dfrac{\operatorname{sen}(x^2+y^2)}{x^2+y^2}}{\dfrac{y^4}{x^2+y^2} + \dfrac{\operatorname{sen}(x^2+y^2)}{x^2+y^2}}$$

$$= \frac{0+1}{0+1} = 1$$