2)[3,5] Seja $f = f(x, y)$ uma função de classe C^2 e seja $g : \mathbb{R}^2 \longrightarrow \mathbb{R}$ dada por

$$g(u, v) = vf(-u + v^2, 2u + v).$$

a) Determine $\frac{\partial g}{\partial u}(u, v)$ e $\frac{\partial g}{\partial v}(u, v)$ em função das derivadas parciais de f.

b) Determine $\frac{\partial^2 g}{\partial u \partial v}(u, v)$ em função das derivadas parciais de f.

c) Calcule $\frac{\partial^2 g}{\partial u \partial v}(3, -2)$, sabendo que

$\nabla f(1, 4) = (5, 3)$, $f(1, 4) = -2$, $\frac{\partial^2 f}{\partial x \partial y}(1, 4) = \frac{\partial^2 f}{\partial x^2}(1, 4) = 1$ e $\frac{\partial^2 f}{\partial y^2}(1, 4) = -1$.

a) Pela Regra da cadeia,

$$\frac{\partial g}{\partial u}(u,v) = v\left[\frac{\partial f}{\partial x}(-u+v^2, 2u+v).(-1) + \frac{\partial f}{\partial y}(-u+v^2, 2u+v)(2)\right].$$

$$\frac{\partial g}{\partial v}(u,v) = f(-u+v^2, 2u+v) +$$

$$+ v\left[\frac{\partial f}{\partial x}(-u+v^2, 2u+v)(2v) + \frac{\partial f}{\partial y}(-u+v^2, 2u+v)\right].$$

b) Novamente, pela Regra da Cadeia ($f \in C^2$),

$$\frac{\partial^2 g}{\partial u \partial v}(u,v) = \frac{\partial}{\partial u}\left(\frac{\partial g}{\partial v}\right) =$$

$$= \frac{\partial f}{\partial x}(-u+v^2, 2u+v)(-1) + \frac{\partial f}{\partial y}(-u+v^2, 2u+v)(2) +$$

$$+ v\left[\frac{\partial^2 f}{\partial x^2}(-u+v^2, 2u+v)(-2v) + \frac{\partial^2 f}{\partial y \partial x}(-u+v^2, 2u+v)(4v) + \right.$$

$$\left. + \frac{\partial^2 f}{\partial x \partial y}(-u+v^2, 2u+v)(-1) + \frac{\partial^2 f}{\partial y^2}(-u+v^2, 2u+v)(2)\right] =$$

$f \in C^2$; pelo Teo de Schwarz

$$\underset{=}{\downarrow}\ 2\frac{\partial f}{\partial y}(-u+v^2, 2u+v) - \frac{\partial f}{\partial x}(-u+v^2, 2u+v) + 2v\frac{\partial^2 f}{\partial y^2}(-u+v^2, 2u+v)$$

$$+ (-2v^2)\frac{\partial^2 f}{\partial x^2}(-u+v^2, 2u+v) + v(4v-1)\frac{\partial^2 f}{\partial x \partial y}(-u+v^2, 2u+v).$$

c) $x(u,v) = -u + v^2$
$y(u,v) = 2u + v$
$u = 3$
$v = -2$

$$\Rightarrow \begin{cases} x(3,-2) = -3+4 = 1 \\ y(3,-2) = 6-2 = 4 \end{cases}$$

B

∴ pelo que foi calculado acima, pelo resultado obtido no ítem b) e sabendo que $\nabla f(1,4) = \left(\frac{\partial f}{\partial x}(1,4), \frac{\partial f}{\partial y}(1,4)\right)$ obtemos que,

$$\frac{\partial^2 g}{\partial u \partial v}(3,-2) = 2 \cdot \overbrace{\frac{\partial f}{\partial y}(1,4)}^{3} - \overbrace{\frac{\partial f}{\partial x}(1,4)}^{5} + (-4) \cdot \overbrace{\frac{\partial^2 f}{\partial y^2}(1,4)}^{-1} +$$

$$+ (-8) \underbrace{\frac{\partial^2 f}{\partial x^2}(1,4)}_{1} + (-2)(-9) \underbrace{\frac{\partial^2 f}{\partial x \partial y}(1,4)}_{1}$$

$$= 6 - 5 + 4 - 8 + 18$$

$$= 28 - 13 = 15$$

$$\boxed{\text{Resp: } \frac{\partial^2 g}{\partial u \partial v}(3,-2) = 15}$$