

Questão A3) (Valor: 2.0)

a) Prove que: $\left| \sqrt{1+x} - \left(1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} \right) \right| \leq \frac{x^4}{16}, \forall x \geq 0.$

b) Prove que $\int_0^1 \sqrt{1+x^4} dx \approx 1 + \frac{1}{2.5} - \frac{1}{8.9} + \frac{1}{13.16}$, com erro, em módulo, inferior a $\frac{1}{200}$.

a) Sejam $f(x) = \sqrt{1+x}$ e $x_0 = 0$.

$$f(x) = (1+x)^{\frac{1}{2}} \Rightarrow f(0) = 1$$

$$f'(x) = \frac{1}{2}(1+x)^{-\frac{1}{2}} \Rightarrow f'(0) = \frac{1}{2}$$

$$f''(x) = -\frac{1}{4}(1+x)^{-\frac{3}{2}} \Rightarrow f''(0) = -\frac{1}{4}$$

$$f'''(x) = \frac{3}{8}(1+x)^{-\frac{5}{2}} \Rightarrow f'''(0) = \frac{3}{8}$$

$$f^{(4)}(x) = -\frac{15}{16}(1+x)^{-\frac{7}{2}}$$

$$\Rightarrow P_3(x) = 1 + \frac{1}{2}x - \frac{1}{4} \cdot \frac{1}{2}x^2 + \frac{3}{8} \cdot \frac{1}{6}x^3 = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16}$$

∃ \bar{x} entre x e x_0 tal que $f(x) - P_3(x) = E(x)$, onde $E(x) = \frac{f^{(4)}(\bar{x})}{4!} x^4 = \frac{-15}{4! \cdot 16} (1+\bar{x})^{-\frac{7}{2}} x^4$

$$\Rightarrow |f(x) - P_3(x)| = |E(x)| = \left| \frac{-15}{24 \cdot 16} (1+\bar{x})^{-\frac{7}{2}} x^4 \right| \leq \frac{x^4}{16}, \text{ pois } \left| \frac{-15}{24} \right| < 1 \text{ e } |1+\bar{x}| < 1, \forall \bar{x} > 0.$$

b) $f(x^4) = \sqrt{1+x^4}$, $P_3(x^4) = 1 + \frac{x^4}{2} - \frac{x^8}{8} + \frac{x^{12}}{16}$, $|E(x^4)| \leq \frac{x^{16}}{16}$

$$\int_0^1 \sqrt{1+x^4} dx \approx \int_0^1 P_3(x^4) dx = \left[x + \frac{x^5}{2.5} - \frac{x^9}{8.9} + \frac{x^{13}}{13.16} \right]_0^1 = 1 + \frac{1}{2.5} - \frac{1}{8.9} + \frac{1}{13.16}$$

$$\left| \int_0^1 \sqrt{1+x^4} dx - \int_0^1 P_3(x^4) dx \right| = \left| \int_0^1 E(x^4) dx \right| \leq \int_0^1 |E(x^4)| dx \leq \int_0^1 \frac{x^{16}}{16} dx = \frac{x^{17}}{16 \cdot 17} \Big|_0^1 = \frac{1}{16 \cdot 17} = \frac{1}{272} < \frac{1}{200}$$