4. (2, 5) Seja $g : \mathbb{R}^2 \to \mathbb{R}$, $g = g(x, y)$, uma função de classe $\mathcal{C}^2$ em $\mathbb{R}^2$. Seja

$$f(t, u) = g(tu, u - 2t).$$

(a) Calcule $\dfrac{\partial^2 f}{\partial t \partial u}(t, u)$ em termos das derivadas parciais de g.

(b) Calcule $\dfrac{\partial^2 f}{\partial t \partial u}(1, 2)$, sabendo que $\dfrac{\partial^2 g}{\partial x^2}(2, 0) = \dfrac{\partial^2 g}{\partial y^2}(2, 0)$ e que $\dfrac{\partial g}{\partial x}(2, 0) = 8$.

a) $f(t,u) = g(x(t,u), y(t,u)) \qquad x(t,u) = tu\ , \ y(t,u) = u - 2t.$

$$\frac{\partial f}{\partial u}(t,u) = \frac{\partial g}{\partial x}(x,y)\cdot\frac{\partial x}{\partial u}(t,u) + \frac{\partial g}{\partial y}(x,y)\frac{\partial y}{\partial u}(t,u) = \frac{\partial g}{\partial x}(x,y)\cdot t + \frac{\partial g}{\partial y}(x,y)\cdot 1$$

$$\frac{\partial^2 f}{\partial t\partial u}(t,u) = \frac{\partial}{\partial t}\left(\frac{\partial g}{\partial x}(x,y)\right)\cdot t + \frac{\partial g}{\partial x}(x,y)\cdot 1 + \frac{\partial}{\partial t}\left(\frac{\partial g}{\partial y}(x,y)\right)$$

$$= \left[\frac{\partial^2 g}{\partial x^2}(x,y)\cdot\frac{\partial x}{\partial t}(t,u) + \frac{\partial^2 g}{\partial y\partial x}(x,y)\cdot\frac{\partial y}{\partial t}(t,u)\right]\cdot t + \frac{\partial g}{\partial x}(x,y)\cdot 1$$

$$+ \frac{\partial^2 g}{\partial x\partial y}(x,y)\cdot\frac{\partial x}{\partial t}(t,u) + \frac{\partial^2 g}{\partial y^2}(x,y)\,\frac{\partial y}{\partial t}(t,u)$$

$$= \frac{\partial^2 g}{\partial x^2}\cdot tu + \frac{\partial^2 g}{\partial y\partial x}(x,y)\cdot(-2t)$$

$$+ \frac{\partial g}{\partial x}(x,y) + \frac{\partial^2 g}{\partial x\partial y}(x,y)\cdot u + \frac{\partial^2 g}{\partial y^2}(x,y)\cdot(-2)$$

Sendo g de classe $\mathcal{C}^2$, $\dfrac{\partial^2 g}{\partial y\partial x}(x,y) = \dfrac{\partial^2 g}{\partial x\partial y}(x,y)$

$$\therefore \frac{\partial^2 f}{\partial t\partial u}(t,u) = \frac{\partial^2 g}{\partial x^2}(x,y)\cdot tu + (u-2t)\frac{\partial^2 g}{\partial x\partial y}(x,y) - 2\frac{\partial^2 g}{\partial y^2}(x,y) + \frac{\partial g}{\partial x}(x,y).$$

b) $t = 1,\ u = 2 \Rightarrow \quad x = 1\cdot 2 = 2\ ,\ y = 2 - 2 = 0.$

$$\therefore \frac{\partial^2 f}{\partial t\partial u}(1,2) = \cancel{\frac{\partial^2 g}{\partial x^2}(2,0)\cdot 2} + \underbrace{\cancel{(2-2)}}_{0}\frac{\partial^2 g}{\partial x\partial y}(2,0) - \cancel{2\frac{\partial^2 g}{\partial y^2}(2,0)} + \frac{\partial g}{\partial x}(2,0) = \boxed{\frac{\partial g}{\partial x}(2,0) = 8}$$