EXAMPLE 2 A curve C is defined by the parametric equations $x = t^2$ and $y = t^3 - 3t$.

(a) Show that C has two tangents at the point $(3, 0)$ and find their equations.
(b) Find the points on C where the tangent is horizontal or vertical.
(c) Determine where the curve rises and falls and where it is concave upward or downward.
(d) Sketch the curve.

SOLUTION

(a) Notice that $y = t^3 - 3t = t(t^2 - 3) = 0$ when $t = 0$ or $t = \pm\sqrt{3}$. Therefore, the point $(3, 0)$ on C arises from two values of the parameter, $t = \sqrt{3}$ and $t = -\sqrt{3}$. This indicates that C crosses itself at $(3, 0)$. Since

$$\frac{dy}{dx} = \frac{\dfrac{dy}{dt}}{\dfrac{dx}{dt}} = \frac{3t^2 - 3}{2t} = \frac{3}{2}\left(t - \frac{1}{t}\right)$$

the slope of the tangent when $t = \pm\sqrt{3}$ is $dy/dx = \pm 6/(2\sqrt{3}) = \pm\sqrt{3}$ so the equations of the tangents at $(3, 0)$ are

$$y = \sqrt{3}\,(x - 3) \qquad \text{and} \qquad y = -\sqrt{3}\,(x - 3)$$

(b) C has a vertical tangent when $dx/dt = 2t = 0$, that is, $t = 0$. The corresponding point on C is $(0, 0)$. C has a horizontal tangent when $dy/dt = 3t^2 - 3 = 0$, that is, $t = \pm 1$. The corresponding points on C are $(1, -2)$ and $(1, 2)$.

(c) Since

$$\frac{dx}{dt} = 2t \qquad \text{and} \qquad \frac{dy}{dt} = 3(t - 1)(t + 1)$$

we can summarize the parameter intervals in which the curve rises and falls in the following table.

	$t < -1$	$-1 < t < 0$	$0 < t < 1$	$t > 1$
dx/dt	−	−	+	+
dy/dt	+	−	−	+
x	←	←	→	→
y	↑	↓	↓	↑
curve	↖	↙	↘	↗

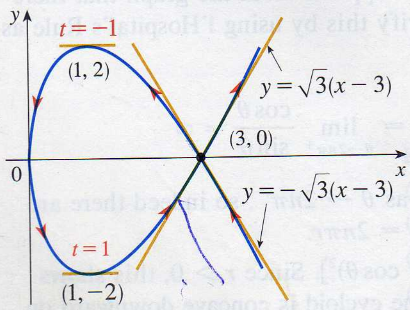

FIGURE 2

To determine concavity we calculate the second derivative:

$$\frac{d^2y}{dx^2} = \frac{\dfrac{d}{dt}\left(\dfrac{dy}{dx}\right)}{\dfrac{dx}{dt}} = \frac{\dfrac{3}{2}\left(1 + \dfrac{1}{t^2}\right)}{2t} = \frac{3(t^2 + 1)}{4t^3}.$$

Thus the curve is concave upward when $t > 0$ and concave downward when $t < 0$.

(d) Using the information from parts (b) and (c), we sketch C in Figure 2.