3. (2,5) Seja $f(x,y)$ uma função de classe C^2 em $\mathbb{R}^2$ (isto é, todas as derivadas parciais até segunda ordem de f são contínuas em todos os pontos de $\mathbb{R}^2$). Seja

$$g(u,v) = f(2uv, u^2 - v^2).$$

(a) Calcule $\dfrac{\partial^2 g}{\partial u \partial v}(u,v)$ em termos das derivadas parciais de f.

(b) Sabendo que

$$\frac{\partial^2 f}{\partial x^2}(x,y) - \frac{\partial^2 f}{\partial y^2}(x,y) = 2 \text{ para todo } (x,y) \in \mathbb{R}^2 \quad (*)$$

e que

$$f(x,0) = x^2 \text{ para todo } x \in \mathbb{R}, \quad (**)$$

calcule $\dfrac{\partial^2 g}{\partial u \partial v}(1,1)$.

(a) $g(u,v) = f(\underbrace{2uv}_{x}, \underbrace{u^2-v^2}_{y})$

$\dfrac{\partial x}{\partial u} = 2v$, $\dfrac{\partial x}{\partial v} = 2u$, $\dfrac{\partial y}{\partial u} = 2u$ e $\dfrac{\partial y}{\partial v} = -2v$

$$\frac{\partial g}{\partial v}(u,v) = \frac{\partial f}{\partial x}(x,y)\,2u + \frac{\partial f}{\partial y}(x,y)(-2v)$$

$$\frac{\partial^2 g}{\partial u \partial v}(u,v) = 2\left[\frac{\partial f}{\partial x}(x,y)\right] + 2u\left[\frac{\partial^2 f}{\partial x^2}(x,y)\,2v + \frac{\partial^2 f}{\partial y \partial x}(x,y)\,2u\right] - 2v\left[\frac{\partial^2 f}{\partial x \partial y}(x,y)\,2v + \frac{\partial^2 f}{\partial y^2}(x,y)\,2u\right]$$

Como f é de classe C^2, $\dfrac{\partial^2 f}{\partial x \partial y}(x,y) = \dfrac{\partial^2 f}{\partial y \partial x}(x,y)$ $\forall (x,y)$.

Logo
$$\boxed{\frac{\partial^2 g}{\partial u \partial v}(u,v) = 2\frac{\partial f}{\partial x}(x,y) + 4uv\left[\frac{\partial^2 f}{\partial x^2}(x,y) - \frac{\partial^2 f}{\partial y^2}(x,y)\right] + (4u^2 - 4v^2)\frac{\partial^2 f}{\partial x \partial y}(x,y).}$$

(b) Se $(u,v) = (1,1)$ então $(x,y) = (2,0)$. Pelo item (a),

$$\frac{\partial^2 g}{\partial u \partial v}(1,1) = 2\frac{\partial f}{\partial x}(2,0) + 4\left[\frac{\partial^2 f}{\partial x^2}(2,0) - \frac{\partial^2 f}{\partial y^2}(2,0)\right] + (4-4)\frac{\partial^2 f}{\partial x \partial y}(2,0)$$

Por (**) temos que $\dfrac{\partial f}{\partial x}(x,0) = 2x$. Logo $\dfrac{\partial f}{\partial x}(2,0) = 4$. Por (*) e (**) temos então que

$$\boxed{\frac{\partial^2 g}{\partial u \partial v}(1,1) = 2.4 + 4.2 = 16}$$