3. (2,5) Seja $f(x,y)$ uma função de classe C^2 em $\mathbb{R}^2$ (isto é, todas as derivadas parciais até segunda ordem de f são contínuas em todos os pontos de $\mathbb{R}^2$). Seja

$$g(u,v) = f(u^2 - v^2, 2uv).$$

(a) Calcule $\dfrac{\partial^2 g}{\partial u \partial v}(u,v)$ em termos das derivadas parciais de f.

(b) Sabendo que

$$\frac{\partial^2 f}{\partial x^2}(x,y) - \frac{\partial^2 f}{\partial y^2}(x,y) = 2 \text{ para todo } (x,y) \in \mathbb{R}^2 \qquad (*)$$

e que

$$f(0,y) = y^2 \text{ para todo } y \in \mathbb{R}, \qquad (**)$$

calcule $\dfrac{\partial^2 g}{\partial u \partial v}(1,1)$.

(a) $g(u,v) = f(\underbrace{u^2-v^2}_{x}, \underbrace{2uv}_{y})$

$$\boxed{\frac{\partial x}{\partial u} = 2u \quad \frac{\partial x}{\partial v} = -2v \quad \frac{\partial y}{\partial u} = 2v \quad \text{e } \frac{\partial y}{\partial v} = 2u}$$

$$\frac{\partial g}{\partial v}(u,v) = \frac{\partial f}{\partial x}(x,y)(-2v) + \frac{\partial f}{\partial y}(x,y)\,2u$$

$$\frac{\partial^2 g}{\partial u \partial v}(u,v) = -2v\left[\frac{\partial^2 f}{\partial x^2}(x,y)(2u) + \frac{\partial^2 f}{\partial y \partial x}(x,y)\,2v\right] + 2\frac{\partial f}{\partial y}(x,y) + 2u\left[\frac{\partial^2 f}{\partial x \partial y}(x,y)\,2u + \frac{\partial^2 f}{\partial y^2}(x,y)\,2v\right]$$

Como f é de classe C^2, $\dfrac{\partial^2 f}{\partial x \partial y}(x,y) = \dfrac{\partial^2 f}{\partial y \partial x}(x,y)$ $\forall (x,y)$.

Logo

$$\boxed{\frac{\partial^2 g}{\partial u \partial v}(u,v) = -4uv\frac{\partial^2 f}{\partial x^2}(x,y) + (4u^2-4v^2)\frac{\partial^2 f}{\partial x \partial y}(x,y) + 4uv\frac{\partial^2 f}{\partial y^2}(x,y) + 2\frac{\partial f}{\partial y}(x,y)}$$

(b) Se $(u,v) = (1,1) \Longrightarrow (x,y) = (0,2)$. Pelo item (a):

$$\frac{\partial^2 g}{\partial u \partial v}(1,1) = 4\left[\frac{\partial^2 f}{\partial y^2}(0,2) - \frac{\partial^2 f}{\partial x^2}(0,2)\right] + (4-4)\frac{\partial^2 f}{\partial x \partial y}(0,2) + 2\frac{\partial f}{\partial y}(0,2).$$

Por (**) temos que $\dfrac{\partial f}{\partial y}(0,y) = 2y$, e daí, $\dfrac{\partial f}{\partial y}(0,2) = 4$. Usando (*), temos

$$\boxed{\frac{\partial^2 g}{\partial u \partial v}(1,1) = 4\cdot(-2) + 2\cdot 4 = 0}$$