2. (3,0) Seja $f(x,y) = \begin{cases} \dfrac{\cos(x^2+y^2)-1}{x^2+y^2} & \text{se } (x,y) \neq (0,0) \\ 0 & \text{se } (x,y) = (0,0) \end{cases}$

(a) Calcule $\dfrac{\partial f}{\partial y}(x,y)$, para todo $(x,y) \in \mathbb{R}^2$.

(b) É $\dfrac{\partial f}{\partial y}$ contínua em $(0,0)$?

(c) Mostre que f é diferenciável em $(0,0)$.

(a) Se $(x,y) \neq (0,0)$,

$$\frac{\partial f}{\partial y}(x,y) = \frac{-2y(x^2+y^2)\operatorname{sen}(x^2+y^2) - 2y[\cos(x^2+y^2)-1]}{(x^2+y^2)^2}$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{h\to 0} \frac{f(0,0+h)-f(0,0)}{h} = \lim_{h\to 0} \frac{\cos h^2 - 1}{h^3} = \text{(L'Hospital)}$$

$$= -\lim_{h\to 0} \frac{2h \operatorname{sen} h^2}{3h^2} = -\frac{2}{3} \lim_{h\to 0} h \cdot \lim_{h\to 0} \frac{\operatorname{sen} h^2}{h^2} = -\frac{2}{3}\cdot 0 \cdot 1 = 0$$

(b) $$\frac{\partial f}{\partial y}(x,y) = -2y \underbrace{\frac{\operatorname{sen}(x^2+y^2)}{x^2+y^2}}_{\xrightarrow{(x,y)\to(0,0)}\ 1} - 2y \underbrace{\frac{\cos(x^2+y^2)-1}{(x^2+y^2)^2}}_{\to\ ?}$$

$$\lim_{(x,y)\to(0,0)} \frac{\cos(x^2+y^2)-1}{(x^2+y^2)^2} = \lim_{u\to 0} \frac{\cos u - 1}{u^2} = -\lim_{u\to 0} \frac{\operatorname{sen} u}{2u} = -\frac{1}{2}$$

Logo $$\lim_{(x,y)\to(0,0)} \frac{\partial f}{\partial y}(x,y) = -2.0.1 - 2.0.\left(-\frac{1}{2}\right) = 0 = \frac{\partial f}{\partial y}(0,0)$$

Logo $\dfrac{\partial f}{\partial y}$ é contínua em $(0,0)$.