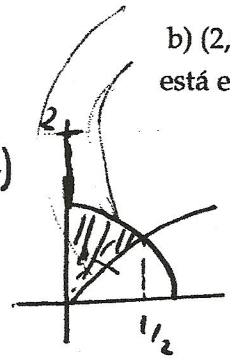
-A- e B

Questão 2. a) (1,5 ponto) Seja $B = \{(x,y) \in \mathbb{R}^2 : 2x^2 + y^2 \le 1 \text{ e } y \ge \sqrt{x}\}$. Calcule o volume do sólido obtido pela rotação de B em torno da reta $y = 2$.

b) (2,0 pontos) Calcule a área da região A formada pelos pontos $(x,y) \in \mathbb{R}^2$ tais que y está entre $f(x) = \sqrt{1-2x^2}$ e $g(x) = \sqrt{x}$, para $0 \le x \le \frac{1}{\sqrt{2}}$.

a)

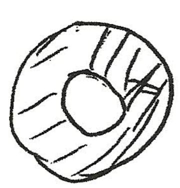


$$2x^2 + y^2 = 1 \xRightarrow{y>0} y = \sqrt{1-2x^2}$$

$$\begin{cases} 2x^2 + y^2 = 1 \\ y = \sqrt{x} \Rightarrow y^2 = x \end{cases} \Rightarrow 2x^2 + x = 1$$

$$2x^2 + x - 1 = 0 \quad \boxed{x = 1/2} \quad \cancel{x = -1}$$

Calculando $A(x)$:

$2 - \sqrt{1-2x^2}$  $2 - \sqrt{x}$

$$A(x) = \pi(2-\sqrt{x})^2 - \pi(2-\sqrt{1-2x^2})^2 = \pi\left[4\sqrt{1-2x^2} + x - 4\sqrt{x} - 1 + 2x^2\right]$$

Logo

$$V = \int_0^{1/2} A(x)\,dx = 4\pi\int_0^{1/2}\sqrt{1-2x^2}\,dx + \pi\int_0^{1/2}(x - 4\sqrt{x} - 1 + 2x^2)\,dx$$

1) $$\int_0^{1/2}\sqrt{1-2x^2}\,dx = \int_0^{\pi/4}\sqrt{1-\operatorname{sen}^2 u}\,\frac{\cos u}{\sqrt{2}}\,du = \frac{1}{\sqrt{2}}\int_0^{\pi/4}\cos^2 u\,du = \frac{1}{\sqrt{2}}\int_0^{\pi/4}\left(\frac{1}{2} + \frac{\cos 2u}{2}\right)du =$$

$x = \frac{\operatorname{sen} u}{\sqrt{2}}$ $\quad -\pi/2 \le u \le \pi/2 \quad \therefore \cos u > 0 \Rightarrow \sqrt{1-\operatorname{sen}^2 u} = \cos u$

$dx = \frac{\cos u}{\sqrt{2}}$ $\quad x = 0 \Rightarrow u = 0 \quad x = \frac{1}{2} \Rightarrow \operatorname{sen} u = \frac{\sqrt{2}}{2} \Rightarrow u = \frac{\pi}{4}$

$$= \frac{1}{\sqrt{2}}\left(\frac{1}{2}u + \frac{\operatorname{sen} 2u}{4}\right)\Big|_0^{\pi/4} = \frac{1}{\sqrt{2}}\left(\frac{\pi}{8} + \frac{1}{4}\right)$$

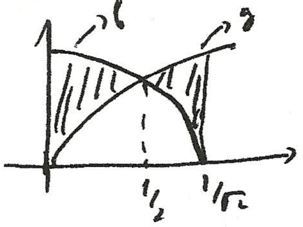
2) $$\int_0^{1/2}[x - 4\sqrt{x} - 1 + 2x^2]\,dx = \frac{x^2}{2} - 4\,\frac{2}{3}x^{3/2} - x + \frac{2x^3}{3}\Big|_0^{1/2} =$$

$$= \frac{1}{8} - \frac{8}{3}\cdot\frac{1}{2\sqrt{2}} - \frac{1}{2} + \frac{2}{3}\cdot\frac{1}{8} = -\frac{7}{24} - \frac{4}{3\sqrt{2}}$$

Logo

$$V = \frac{4\pi}{\sqrt{2}}\left(\frac{\pi}{8} + \frac{1}{4}\right) + \pi\left[-\frac{7}{24} - \frac{4}{3\sqrt{2}}\right]$$

b)



$f(x) = \sqrt{1-2x^2}$

$g(x) = \sqrt{x}$

Pelo item (a), $f(x) = g(x) \overset{x>0}{\Longrightarrow} x = \frac{1}{2}$

Logo

$$A = \int_0^{1/2} \sqrt{1-2x^2}\,dx - \int_0^{1/2} \sqrt{x}\,dx + \int_{1/2}^{\frac{1}{\sqrt{2}}} \sqrt{x}\,dx - \int_{1/2}^{1/\sqrt{2}} \sqrt{1-2x^2}\,dx$$

Calculando as integrais:

I) $\displaystyle\int_0^{1/2} \sqrt{1-2x^2}\,dx = \frac{1}{\sqrt{2}}\left(\frac{\pi}{8} + \frac{1}{4}\right)$ (item (a))

II) $\displaystyle\int_0^{1/2} \sqrt{x}\,dx = \frac{2}{3} x^{3/2}\Big|_0^{1/2} = \frac{2}{3} \cdot \frac{1}{2\sqrt{2}} = \frac{1}{3\sqrt{2}}$

III) $\displaystyle\int_{1/2}^{1/\sqrt{2}} \sqrt{x}\,dx = \frac{2}{3} x^{3/2}\Big|_{1/2}^{1/\sqrt{2}} = \frac{2}{3}\left[\frac{1}{\sqrt[4]{8}} - \frac{1}{2\sqrt{2}}\right]$

IV) $\displaystyle\int_{1/2}^{1/\sqrt{2}} \sqrt{1-2x^2}\,dx = \int_{\pi/4}^{\pi/2} \sqrt{1-\operatorname{sen}^2 u}\,\frac{\cos u}{\sqrt{2}}\,du = \frac{1}{\sqrt{2}}\int_{\pi/4}^{\pi/2} \cos^2 u\,du =$

$x = \dfrac{\operatorname{sen} u}{\sqrt{2}}$ $\quad -\pi/2 \leq u \leq \pi/2 \quad \therefore \cos u > 0$

$dx = \dfrac{\cos u}{\sqrt{2}}$ $\quad x = 1/2 \Rightarrow \operatorname{sen} u = \frac{\sqrt{2}}{2} \Rightarrow u = \pi/4$

$\quad x = \frac{1}{\sqrt{2}} \Rightarrow \operatorname{sen} u = 1 \Rightarrow u = \pi/2$

$$= \frac{1}{\sqrt{2}}\left[\frac{1}{2}u + \frac{\operatorname{sen} 2u}{4}\right]\Bigg|_{\pi/4}^{\pi/2} = \frac{1}{\sqrt{2}}\left[\frac{1}{2}\frac{\pi}{2} + 0 - \frac{1}{2}\frac{\pi}{4} - \frac{1}{4}\right] = \frac{1}{\sqrt{2}}\left[\frac{\pi}{4} - \frac{1}{4}\right]$$

Logo

$$A = \frac{1}{\sqrt{2}}\left(\frac{\pi}{8} + \frac{1}{4}\right) - \frac{1}{3\sqrt{2}} + \frac{2}{3}\left[\frac{1}{\sqrt[4]{8}} - \frac{1}{2\sqrt{2}}\right] - \frac{1}{\sqrt{2}}\left[\frac{\pi}{4} - \frac{1}{4}\right].$$