A

Questão 4.

(1,0) a) Mostre que para todo $x \in \mathbb{R}$ temos

$$\left|\text{sen}(x^4) - (x^4 - \frac{x^{12}}{3!} + \frac{x^{20}}{5!})\right| \leq \frac{|x|^{24}}{6!}$$

Sejam $f(y) = \text{sen}(y)$ e $y_0 = 0$.
O polinômio de Taylor de f de grau 5, em torno de y_0 é dado por:

$$p_5(y) = f(0)+f'(0)(y-0)+\frac{f''(0)}{2!}(y-0)^2+\frac{f'''(0)}{3!}(y-0)^3+\frac{f^{(4)}(0)}{4!}(y-0)^4+\frac{f^{(5)}(0)}{5!}(y-0)^5,$$

com $f(0) = \text{sen}(0) = 0$, $f'(0) = \cos(0) = 1$, $f''(y_0) = -\text{sen}(0) = 0$, $f'''(0) = -\cos(0) = -1$, $f^{(4)}(0) = \text{sen}(0) = 0$ e $f^{(5)}(0) = \cos(0) = 1$.

Ou seja,

$$p_5(y) = y - \frac{y^3}{3!} + \frac{y^5}{5!}.$$

Considere $y = x^4$, segundo a fórmula de Taylor, existe um $\bar{x}$ entre 0 e x^4 tal que: $f(x^4) = p_5(x^4) + E(x^4)$, onde $E(y) = \dfrac{f^{(6)}(\bar{x})}{6!}y^6$. Ou seja,

$$\left|\text{sen}(x^4) - (x^4 - \frac{x^{12}}{3!} + \frac{x^{20}}{5!})\right| = \left|\frac{f^{(6)}(\bar{x})}{6!}(x^4)^6\right| \leq \frac{|x|^{24}}{6!}, \text{ pois } |f^{(6)}(\bar{x})| = |-\text{sen}(\bar{x})| \leq 1$$

(1,0) b) Avalie $\displaystyle\int_0^1 \text{sen}(x^4)dx$ com erro inferior a $\dfrac{1}{2^{12}}$.

$$\left|\int_0^1 \text{sen}(x^4)dx - \int_0^1 p_n(x^4)dx\right| = \left|\int_0^1 \left(\text{sen}(x^4) - p_n(x^4)\right)dx\right| \leq \int_0^1 \left|\text{sen}(x^4) - p_n(x^4)\right|dx$$
$$\leq \int_0^1 \left|f^{(n+1)}(\bar{x})\right| \frac{x^{4n+4}}{(n+1)!}dx \leq \int_0^1 \frac{x^{4n+4}}{(n+1)!}dx = \left[\frac{x^{4n+5}}{(n+1)!(4n+5)}\right]_0^1 = \frac{1}{(n+1)!(4n+5)}$$
$$< \frac{1}{2^{12}} \Longrightarrow n = 5.$$

$\Big($Observe que $\left|f^{(n+1)}(\bar{x})\right| \leq 1$, pois $\left|f^{(n+1)}(\bar{x})\right| = |\cos(\bar{x})|$ ou $\left|f^{(n+1)}(\bar{x})\right| = |\text{sen}(\bar{x})|\Big)$.

$$\int_0^1 \text{sen}(x^4)dx \approx \int_0^1 p_5(x^4)dx = \int_0^1 \left(x^4 - \frac{x^{12}}{3!} + \frac{x^{20}}{5!}\right)dx = \left[\frac{x^5}{5} - \frac{x^{13}}{13.3!} + \frac{x^{21}}{21.5!}\right]_0^1$$
$$= \frac{1}{5} - \frac{1}{13.3!} + \frac{1}{21.5!} = \frac{1}{5} - \frac{1}{78} + \frac{1}{2520} = 0,2 - 0,012821 + 0,000397 = 0,187576.$$