Gabarito da Questão 1 - Prova Tipo B / 2010

Calcule as seguintes integrais indefinidas:

(1,5) *a*) $\int \frac{x^5}{\sqrt{2+x^2}}dx$

Fazendo a mudança de variáveis $x = \sqrt{2}\tan u$, temos que $dx = \sqrt{2}\sec^2 u du$, logo

$$\begin{aligned}
\int \frac{x^5}{\sqrt{2+x^2}}dx &= 4\sqrt{2}\int \tan^5 u \sec u du \\
&= 4\sqrt{2}\int (\sec^2 u - 1)^2(\sec u \tan u)du \\
&= 4\sqrt{2}\int (\sec^4 u - 2\sec^2 u + 1)(\sec u \tan u)du \\
&= 4\sqrt{2}\left(\frac{\sec^5 u}{5} - 2\frac{\sec^3 u}{3} + \sec u\right) + C \\
&= 4\sqrt{2}\left\{\frac{1}{5}\left(1+\frac{x^2}{2}\right)^{5/2} - \frac{2}{3}\left(1+\frac{x^2}{2}\right)^{3/2} + \left(1+\frac{x^2}{2}\right)^{1/2}\right\} + C \\
&= \frac{1}{5}(2+x^2)^{5/2} - \frac{4}{3}(2+x^2)^{3/2} + 4(2+x^2)^{1/2} + C\,.
\end{aligned}$$

(2,0) *b*) $\int \frac{6e^x}{(e^x+4)(e^{2x}+4e^x+6)}dx$

Fazendo a mudança de variáveis $e^x = u$, temos que $e^x dx = du$, logo

$$\int \frac{6e^x}{(e^x+4)(e^{2x}+4e^x+6)}dx = \int \frac{6}{(u+4)(u^2+4u+6)}dx\,. \qquad (1)$$

Vamos agora encontrar $A, B, C \in \mathbb{R}$ tais que

$$\frac{6}{(u+4)(u^2+4u+6)} = \frac{A}{u+4} + \frac{Bu+C}{u^2+4u+6}\,.$$

Isso implica que A, B, C satisfazem o sistema linear

$$\begin{cases} A + B = 0 \\ 4A + 4B + C = 0 \\ 6A + 4C = 6 \end{cases},$$

e portanto, $A = 1$, $B = -1$ e $C = 0$. Voltando à equação (1), temos que

$$\begin{aligned}
\int \frac{6e^x}{(e^x+4)(e^{2x}+4e^x+6)}dx &= \int \left(\frac{1}{u+4} - \frac{u}{u^2+4u+6}\right)du \\
&= \ln|u+4| - \frac{1}{2}\int \frac{u}{\left(\frac{u+2}{\sqrt{2}}\right)^2+1}du \\
&= \ln|u+4| - \frac{1}{\sqrt{2}}\int \frac{\sqrt{2}v-2}{v^2+1}dv\,, \quad \text{onde } v = \frac{u+2}{\sqrt{2}} \\
&= \ln|u+4| - \frac{1}{2}\ln(1+v^2) + \sqrt{2}\arctan v + C \\
&= \ln(e^x+4) - \frac{1}{2}\ln\left(1+\left(\frac{e^x+2}{\sqrt{2}}\right)^2\right) + \sqrt{2}\arctan\left(\frac{e^x+2}{\sqrt{2}}\right) + C
\end{aligned}$$