Gabarito da Questão 1 - Prova Tipo A / 2010

Calcule as seguintes integrais indefinidas:

$(1,5)$ $a)$ $\int \frac{x^5}{\sqrt{3+x^2}}dx$

Fazendo a mudança de variáveis $x = \sqrt{3}\tan u$, temos que $dx = \sqrt{3}\sec^2 u du$, logo

$$\begin{aligned}
\int \frac{x^5}{\sqrt{3+x^2}}dx &= 9\sqrt{3}\int \tan^5 u \sec u du \\
&= 9\sqrt{3}\int (\sec^2 u - 1)^2(\sec u \tan u)du \\
&= 9\sqrt{3}\int (\sec^4 u - 2\sec^2 u + 1)(\sec u \tan u)du \\
&= 9\sqrt{3}\left(\frac{\sec^5 u}{5} - 2\frac{\sec^3 u}{3} + \sec u\right) + C \\
&= 9\sqrt{3}\left\{\frac{1}{5}\left(1+\frac{x^2}{3}\right)^{5/2} - \frac{2}{3}\left(1+\frac{x^2}{3}\right)^{3/2} + \left(1+\frac{x^2}{3}\right)^{1/2}\right\} + C \\
&= \frac{1}{5}(3+x^2)^{5/2} - 2(3+x^2)^{3/2} + 9(3+x^2)^{1/2} + C\,.
\end{aligned}$$

$(2,0)$ $b)$ $\int \frac{3e^x}{(e^x+2)(e^{2x}+2e^x+3)}dx$

Fazendo a mudança de variáveis $e^x = u$, temos que $e^x dx = du$, logo

$$\int \frac{3e^x}{(e^x+2)(e^{2x}+2e^x+3)}dx = \int \frac{3}{(u+2)(u^2+2u+3)}dx\,. \tag{1}$$

Vamos agora encontrar $A, B, C \in \mathbb{R}$ tais que

$$\frac{3}{(u+2)(u^2+2u+3)} = \frac{A}{u+2} + \frac{Bu+C}{u^2+2u+3}\,.$$

Isso implica que A, B, C satisfazem o sistema linear

$$\begin{cases} A + B = 0 \\ 2A + 2B + C = 0 \\ 3A + 2C = 3 \end{cases},$$

e portanto, $A = 1$, $B = -1$ e $C = 0$. Voltando à equação (1), temos que

$$\begin{aligned}
\int \frac{3e^x}{(e^x+2)(e^{2x}+2e^x+3)}dx &= \int \left(\frac{1}{u+2} - \frac{u}{u^2+2u+3}\right)du \\
&= \ln|u+2| - \frac{1}{2}\int \frac{u}{\left(\frac{u+1}{\sqrt{2}}\right)^2 + 1}du \\
&= \ln|u+2| - \frac{1}{\sqrt{2}}\int \frac{\sqrt{2}v - 1}{v^2+1}dv\,, \quad \text{onde } v = \frac{u+1}{\sqrt{2}} \\
&= \ln|u+2| - \frac{1}{2}\ln(1+v^2) + \frac{1}{\sqrt{2}}\arctan v + C \\
&= \ln(e^x+2) - \frac{1}{2}\ln\left(1 + \left(\frac{e^x+1}{\sqrt{2}}\right)^2\right) + \frac{1}{\sqrt{2}}\arctan\left(\frac{e^x+1}{\sqrt{2}}\right) + C
\end{aligned}$$