Questão 2. Seja $f:]-1,1[\longrightarrow \mathbb{R}$ uma função tal que $f(0)=0$ e

$$\frac{x^6}{\cos x^2 - 1} \le f(x) \le \frac{x^4}{x^2+\operatorname{sen} x^2}, \quad \forall x \neq 0.$$

(2,0) a) Verifique que f é derivável em $x_0 = 0$. Justifique.

(1,0) b) Seja $g(x) = [f(3x)+x]\operatorname{arctg}(\sqrt{1-x^2})$. Calcule $g'(0)$.

(a) Temos que mostrar que existe $f'(0) = \lim_{x\to 0} \frac{f(x)-f(0)}{x-0} =$

$= \lim_{x\to 0} \frac{f(x)}{x}$.

Note que

(1) $\lim_{x\to 0} \frac{x^{\cancel{6}\,5}}{\cos(x^2)-1}\cdot\frac{1}{\cancel{x}} = \lim_{x\to 0} \frac{x^5}{-\operatorname{sen}^2(x^2)}\cdot(1+\cos(x^2)) =$

$= \lim_{x\to 0} \underbrace{x}_{\to 0}\cdot \underbrace{\frac{(x^2)^2}{\operatorname{sen}^2(x^2)}}_{\to 1}\cdot\underbrace{(1+\cos(x^2))}_{\to 2} = 0\cdot 1\cdot 2 = 0$

(obs: $\lim_{x\to 0} \frac{\operatorname{sen}(x^2)}{x^2} = \lim_{\substack{u\to 0\\ u=x^2}} \frac{\operatorname{sen} u}{u} = 1$, logo $\lim_{x\to 0}\frac{(x^2)^2}{\operatorname{sen}^2(x^2)} = \lim_{x\to 0}\frac{1}{\left(\frac{\operatorname{sen} x^2}{x^2}\right)^2}$

(2) $\lim_{x\to 0} \frac{x^{\cancel{4}\,3}}{x^2+\operatorname{sen}(x^2)}\cdot\frac{1}{\cancel{x}} = \lim_{x\to 0} \underbrace{x}_{\to 0}\cdot\frac{1}{1+\underbrace{\frac{\operatorname{sen}(x^2)}{x^2}}_{\to 1}} = 0\cdot\frac{1}{2} = 0.$

Como, para $x\neq 0$, temos $\frac{x^6}{\cos(x^2)-1} \le f(x) \le \frac{x^4}{x^2+\operatorname{sen}(x^2)}$, segue q.

- <u>para</u> $x>0$: $\underbrace{\frac{x^6}{\cos(x^2)-1}\cdot\frac{1}{x}}_{\to 0} \le \frac{f(x)}{x} \le \underbrace{\frac{x^4}{x^2+\operatorname{sen}(x^2)}\cdot\frac{1}{x}}_{\to 0}$

Logo, por (1) e (2), e pelo T. do Confronto, temos que

$\lim_{x\to 0^+} \frac{f(x)}{x} = 0$

• <u>para</u> $\underline{x<0}$; $\underbrace{\frac{x^4}{x^2+\operatorname{sen}(x^2)}\cdot\frac{1}{x}}_{\downarrow \atop 0} < \frac{f(x)}{x} < \underbrace{\frac{x^6}{\cos(x^2)-1}\cdot\frac{1}{x}}_{\downarrow \atop 0}$.

Novamente, por (1) e (2), e pelo teorema do Confronto, segue que $\lim_{x\to 0^-}\frac{f(x)}{x}=0$

Portanto $\lim_{x\to 0}\frac{f(x)}{x}=0$, o que implica que f é derivável no 0 e que $f'(0)=0$.

(b) $g(x)=[f(3x)+x].\operatorname{arctg}(\sqrt{1-x^2})$

$\therefore\ g'(x)=[f'(3x).3+1].\operatorname{arctg}(\sqrt{1-x^2})+$

$+[f(3x)+x].\frac{1}{1+(\sqrt{1-x^2})^2}\cdot\frac{1}{\cancel{2}}\frac{1}{\sqrt{1-x^2}}\cdot\cancel{2}x=$

$=[3f'(3x)+1]\operatorname{arctg}(\sqrt{1-x^2})+[f(3x)+x].\frac{-x}{(2-x^2).\sqrt{1-x^2}}$

Logo

$g'(0)=[3\overset{0}{\overset{\shortparallel}{f'(0)}}+1]\operatorname{arctg}(1)+[f(0)+0].0=$

$=\operatorname{arctg}1=\pi/4$