Questão 2. Seja $f :]-1,1[\longrightarrow \mathbb{R}$ uma função tal que $f(0) = 0$ e

$$\frac{x^6}{\cos x^2 - 1} \leq f(x) \leq \frac{x^4}{x^2 + \operatorname{sen} x^2}, \quad \forall x \neq 0.$$

(2,0) a) Verifique que f é derivável em $x_0 = 0$. Justifique.

(1,0) b) Seja $g(x) = [f(2x) + x] \operatorname{arctg}(\sqrt{1-x^2})$. Calcule $g'(0)$.

(a) Temos que mostrar que existe $f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} =$

$= \lim_{x \to 0} \frac{f(x)}{x}$.

Note que:

(1) $\lim_{x \to 0} \frac{x^{\cancel{6}\,5}}{\cos(x^2) - 1} \cdot \frac{1}{\cancel{x}} = \lim_{x \to 0} \frac{x^5}{-\operatorname{sen}^2(x^2)} \cdot (1 + \cos(x^2)) =$

$= \lim_{x \to 0} \underbrace{-x}_{\to 0} \cdot \underbrace{\frac{(x^2)^2}{\operatorname{sen}^2(x^2)}}_{\to 1} \cdot \underbrace{(1 + \cos(x^2))}_{\to 2} = 0 \cdot 1 \cdot 2 = 0$

(obs: $\lim_{x \to 0} \frac{\operatorname{sen}(x^2)}{x^2} = \lim_{u \to 0} \frac{\operatorname{sen} u}{u} = 1$, logo $\lim_{x \to 0} \frac{(x^2)^2}{\operatorname{sen}^2(x^2)} = \lim_{x \to 0} \frac{1}{\left(\frac{\operatorname{sen} x^2}{x^2}\right)^2} =$
$u = x^2$

(2) $\lim_{x \to 0} \frac{x^{\cancel{4}\,3}}{x^2 + \operatorname{sen}(x^2)} \cdot \frac{1}{\cancel{x}} = \lim_{x \to 0} x \cdot \frac{1}{1 + \underbrace{\frac{\operatorname{sen}(x^2)}{x^2}}_{\to 1}} = 0 \cdot \frac{1}{2} = 0$

Como, para $x \neq 0$, temos $\frac{x^6}{\cos(x^2) - 1} \leq f(x) \leq \frac{x^4}{x^2 + \operatorname{sen}(x^2)}$, segue que:

· para $x > 0$: $\underbrace{\frac{x^6}{\cos(x^2) - 1} \cdot \frac{1}{x}}_{\to 0} \leq \frac{f(x)}{x} \leq \underbrace{\frac{x^4}{x^2 + \operatorname{sen}(x^2)} \cdot \frac{1}{x}}_{\to 0}$.

Logo, por (1) e (2), e pelo T. do Confronto, temos que

$\lim_{x \to 0^+} \frac{f(x)}{x} = 0$

• <u>para</u> $\underline{x<0}$: $\underbrace{\dfrac{x^4}{x^2+\operatorname{sen}(x^2)}\cdot\dfrac{1}{x}}_{\to 0} < \dfrac{f(x)}{x} < \underbrace{\dfrac{x^6}{\cos(x^2)-1}\cdot\dfrac{1}{x}}_{\to 0}$

Novamente, por (1) e (2), e pelo T. do Confronto segue que $\lim\limits_{x\to 0^-}\dfrac{f(x)}{x}=0$

Portanto $\lim\limits_{x\to 0}\dfrac{f(x)}{x}=0$, o que implica que f é derivável no 0 e que $f'(0)=0$.

(b) $g(x)=[f(2x)+x]\cdot \operatorname{arctg}(\sqrt{1-x^2})$

$\therefore\ g'(x)=[f'(2x).2+1]\operatorname{arctg}(\sqrt{1-x^2})\ +$

$$[f(2x)+x]\cdot\frac{1}{1+(\sqrt{1-x^2})^2}\cdot\frac{1}{\cancel{2}}\cdot\frac{1}{\sqrt{1-x^2}}\cdot -\cancel{2}x =$$

$$=[2f'(2x)+1]\cdot\operatorname{arctg}(\sqrt{1-x^2})+[f(2x)+x]\cdot\frac{-x}{(2-x^2)\sqrt{1-x^2}}$$

Logo

$$g'(0)=[3.\overset{0}{\overset{\shortparallel}{f'(0)}}+1]\operatorname{arctg}1+[f(0)+0].0=$$

$$=\operatorname{arctg}1=\pi/4.$$