ON SIZE, CIRCUMFERENCE AND CIRCUIT REMOVAL IN 3–CONNECTED MATROIDS

MANOEL LEMOS AND JAMES OXLEY

ABSTRACT. This paper proves several extremal results for 3-connected matroids: a 3-connected matroid M of rank at least six has circumference $c(M)$ at least six and, provided $|E(M)| \geq 4r(M) - 5$, has a circuit whose deletion leaves a 3-connected matroid; if M has a basis B such that $M \backslash e$ is not 3-connected for all e in $E(M) - B$, then $|E(M)| \leq 3r(M) - 4$; and if M is minimally 3-connected but not hamiltonian, then $|E(M)| \leq 3r(M) - c(M)$.

1. INTRODUCTION

Let M be a matroid and A be a subset of $E(M)$. Lemos and Oxley [7] and Lemos, Oxley, and Reid [8] considered the problem of finding a sharp upper bound on $|E(M') - A|$ where M' is a 3-connected minor of M that is minimal with the property that $M|A = M'|A$. The following theorem, the main result of [7], solves this problem in the case when A spans M. Let $\lambda_1(A, M)$ denote the number of connected components of $M|A$. Now $M|A$ can be constructed from a collection $\Lambda_2(A, M)$ of 3-connected matroids by using the operations of direct sum and 2-sum. It follows from work of Cunningham and Edmonds [1] that $\Lambda_2(A, M)$ is unique up to isomorphism. We denote by $\lambda_2(A, M)$ the number of matroids in $\Lambda_2(A, M)$ that are not isomorphic to $U_{1,3}$, the three-element cocircuit.

1.1. Theorem. *Let M be a 3–connected matroid other than $U_{1,3}$ and let A be a non-empty spanning subset of $E(M)$. If M has no proper 3–connected minor M' such that $M'|A = M|A$, then*

$$|E(M)| \leq |A| + \lambda_1(A, M) + \lambda_2(A, M) - 2,$$

unless A is a circuit of size at least four, in which case,

$$|E(M)| \leq 2|A| - 2.$$

It was also shown in [7] that, in a strong sense, the bound in this theorem is best-possible: given a simple matroid N having at least one circuit, there is a minor-minimal 3-connected matroid M for which $M|E(N) = N$ such that $|E(M)|$ attains the bound in the theorem. Therefore the theorem is best-possible for every restriction $M|A$ for which $r^*(M|A) \neq 0$. The first new result of this paper sharpens Theorem 1.1 in the case that A is a basis of M.

1.2. Theorem. *Let M be a 3-connected matroid of rank at least four and B be a basis of M. If M has no proper 3-connected minor that contains B and has B as a basis, then $|E(M)| \leq 3r(M) - 4$.*

1991 *Mathematics Subject Classification.* 05B35.

A *minimally 3-connected matroid* is a 3-connected matroid for which no single-element deletion is 3-connected. Generalizing a result of Halin [3, Satz 7.6] for graphs, Oxley [10, Theorem 4.7] proved the following bound on the size of a minimally 3-connected matroid and characterized the matroids attaining equality in this bound.

1.3. Theorem. *Let M be a minimally 3-connected matroid of rank at least three. Then*

$$|E(M)| \leq \begin{cases} 2r(M), & \text{if } r(M) \leq 6; \\ 3r(M) - 6, & \text{if } r(M) \geq 7. \end{cases}$$

Our second theorem uses Theorem 1.1 to derive a new bound on the size of a minimally 3-connected matroid. The *circumference* $c(M)$ of a matroid that is not free is the maximum size of a circuit of M.

1.4. Theorem. *Let M be a minimally 3-connected matroid. Then*

$$|E(M)| \leq \begin{cases} 2r(M), & \text{if } M \text{ has a hamiltonian circuit;} \\ 3r(M) - c(M), & \text{otherwise.} \end{cases}$$

A comparison of the last two results prompts one to seek a lower bound on the circumference of a 3-connected matroid. The following such bound is obtained in Section 3, and an immediate consequence of this bound is that Theorem 1.4 sharpens Theorem 1.3.

1.5. Theorem. *For a 3-connected matroid M of rank at least six, $c(M) \geq 6$.*

Mader [9] showed that every k-connected simple graph G with minimum degree at least $k+2$ has a cycle C such that $G\backslash C$, the graph obtained from G by deleting the edges of C, is k-connected. Jackson [4] extended this result for 2-connected graphs. In [5], Lemos and Oxley proved that if M is a 2-connected matroid for which

$$|E(M)| \geq 2r(M) + 2 + \max\{0, 1 + r(M) - c(M)\},$$

then M must have a circuit C such that $M\backslash C$ is 2-connected. Moreover, if $|E(M)| \geq 3r(M)$, they showed that this circuit can be chosen so as to avoid some arbitrarily chosen basis B of M. In Section 4, we prove the corresponding results in the 3-connected case:

1.6. Theorem. *Let M be a 3-connected matroid such that $r(M) \geq 6$. If*

$$|E(M)| \geq 3r(M) + 1 + \max\{0, r(M) - c(M)\},$$

then M has a circuit C such that $M\backslash C$ is 3-connected.

1.7. Theorem. *Suppose that M is a 3-connected matroid such that $r(M) \geq 4$ and let B be a basis of M. If*

$$|E(M)| \geq 4r(M) - 3,$$

then M has a circuit C such that $M\backslash C$ is 3-connected and $C \cap B = \emptyset$.

The terminology used here will follow Oxley [11].

2. A bound to the size of a minimally 3-connected matroid

In this section, we shall prove Theorems 1.2 and 1.4, a best-possible bound on the size of a minimal 3-connected matroid that maintains a fixed set as a basis, and a new bound on the size of a minimally 3-connected matroid as a function of its rank and circumference. Both of these results are consequences of Theorem 1.1. For a 3-connected matroid M and a subset A of its ground set, (M, A) is called a *minimal pair* if M has no 3-connected minor M' for which $M'|A = M|A$.

The following lemma plays a key role in the proofs of both theorems. For a basis B of a matroid M and an element e of $E(M) - B$, the unique circuit of M that is contained in $B \cup e$ is denoted by $C(e, B)$.

2.1. Lemma. *Suppose that $(M, B \cup e)$ is a minimal pair and that $r(M) \neq 0$. If B is a basis of M and $e \in E(M) - B$, then*

$$|E(M)| \leq \begin{cases} 2r(M), & \text{if } C(e,B) \text{ is a hamiltonian circuit of } M; \\ 3r(M) - |C(e,B)|, & \text{otherwise.} \end{cases}$$

Proof. Suppose first that $C(e, B)$ is a hamiltonian circuit of M. If $r(M) \in \{1, 2\}$, then $E(M) = B \cup e$ and the result holds. Thus we may assume that $r(M) \geq 3$. Then, by applying Theorem 1.1 to the minimal pair $(M, B \cup e)$, we get that $|E(M)| \leq 2|C(e, B)| - 2 = 2r(M)$, and the result follows.

We may now suppose that $C(e, B)$ is not a hamiltonian circuit of M. Observe that $M|(B \cup e)$ has $C(e, B)$ as a connected component and $B - C(e, B)$ as a non-empty set of coloops. Therefore

$$\lambda_1(B \cup e, M) = 1 + |B - C(e,B)| = 1 + (|B| - |C(e,B)| + 1).$$

Thus $\lambda_1(B \cup e, M) = r(M) + 2 - |C(e, B)|$. We also have that

$$\begin{aligned} \lambda_2(B \cup e, M) &= \lambda_2(C(e,B), M) + |B - C(e,B)| \\ &= (|C(e,B)| - 2) + (|B| - |C(e,B)| + 1) = r(M) - 1. \end{aligned}$$

Hence, by Theorem 1.1,

$$\begin{aligned} |E(M)| &\leq |B \cup e| + \lambda_1(B \cup e, M) + \lambda_2(B \cup e, M) - 2 \\ &= (r(M) + 1) + (r(M) + 2 - |C(e,B)|) + (r(M) - 1) - 2 \\ &= 3r(M) - |C(e,B)|. \end{aligned}$$

□

Theorem 1.4 is a straightforward consequence of this lemma.

Proof of Theorem 1.4. Let C be a circuit of M such that $|C| = c(M)$. Choose an element e of C and a basis B of M such that $C - e \subseteq B$. Clearly $C = C(e, B)$. The result follows immediately from Lemma 2.1. □

Lemos and Oxley [6] proved the analogue of the last theorem for minimally 2-connected matroids, namely, if M is a minimally 2-connected matroid, then

$$|E(M)| \leq 2r(M) + 2 - c(M).$$

A matroid is *hamiltonian* if it has a spanning circuit. Evidently the circumference of such a matroid is one more than its rank. An interesting aspect of Theorem 1.4 is that the wheels and whirls are hamiltonian matroids that are extremal examples for the theorem. Moreover, for each r and each c such that $6 \leq c \leq r$, we shall now describe an extremal example M for this theorem that has rank r and circumference c. Let $\{v_1, v_2, v_3\}$ be one of the vertex classes in the bipartition of $K_{3,r+3-c}$. Let

G be obtained from $K_{3,r+3-c}$ as follows: add a path of length $c-4$ to $K_{3,r+3-c}$ that links v_1 and v_3 but is otherwise disjoint from $K_{3,r+3-c}$; then add a new edge joining v_2 to every vertex of this path other than the ends. It is not difficult to check that the cycle matroid M of G has circumference c, rank r, and has $3r-c$ elements.

Next we shall prove Theorem 1.2 giving a best-possible upper bound on $|E(M)|$ for a minimal pair (M, A) when A is a basis of M. In this theorem, observe that the bound on the rank of M in the hypothesis cannot be lowered: take M to be the rank-3 wheel and let B be its set of spokes. Then $|E(M)| = 6$, but $3r(M) - 4 = 5$.

Proof of Theorem 1.2. First observe that $(M, B \cup e)$ is a minimal pair for all e in $E(M) - B$. Suppose that $C(e, B)$ is hamiltonian for some e in $E(M) - B$. Then, by Lemma 2.1, $|E(M)| \leq 2r(M)$ and so, as $r(M) \geq 4$, we have $|E(M)| \leq 3r(M) - 4$. Thus we may assume that $C(e, B)$ is non-hamiltonian for all e. In that case, by Lemma 2.1 again, $|E(M)| \leq 3r(M) - |C(e, B)|$ for all e. Hence we may assume that

$$|C(e, B)| = 3 \text{ for every element } e \text{ of } E(M) - B. \tag{1}$$

Next we shall prove that the theorem holds unless

$$C(e, B) \cap C(f, B) \neq \emptyset \text{ whenever } e \neq f. \tag{2}$$

Suppose that $e \neq f$ and that $C(e, B) \cap C(f, B) = \emptyset$. In this case, we shall consider the minimal pair $(M, B \cup \{e, f\})$. Observe that $M|(B \cup \{e, f\})$ has $C(e, B), C(f, B)$, and each individual element of $B - [C(e, B) \cup C(f, B)]$ as its connected components. Thus, by (1),

$$\lambda_1(B \cup \{e, f\}, M) = r(M) - 2 \text{ and } \lambda_2(B \cup \{e, f\}, M) = r(M) - 2.$$

Hence, by Theorem 1.1,

$$|E(M)| \leq |B \cup \{e, f\}| + (r(M) - 2) + (r(M) - 2) - 2 = 3r(M) - 4$$

and the result follows. Thus we may suppose that (2) holds.

Assume that there are distinct elements e, f, g in $E(M) - B$ such that

$$C(e, B) = \{e, a, b\}, C(f, B) = \{f, b, c\}, C(g, B) = \{g, c, a\}. \tag{3}$$

In this case, we shall obtain a contradiction. We may assume that $E(M) - (B \cup \{e, f, g\})$ contains an element h otherwise the theorem certainly holds. By (2), $C(h, B)$ must intersect all of $C(e, B)$, $C(f, B)$, and $C(g, B)$. Hence $C(h, B) - h \subseteq \{a, b, c\}$. Thus $\{a, b, c\}$ spans $E(M) - B$. Moreover, $B - \{a, b, c\}$ is non-empty since $r(M) \geq 4$. Hence $\{B - \{a, b, c\}, (E(M) - B) \cup \{a, b, c\}\}$ is a 1-separation of the 3-connected matroid M; a contradiction. Thus (3) cannot occur.

We deduce that B has an element b such that $b \in C(e, B)$ for every e in $E(M) - B$. Let b_e be the element of $C(e, B) - \{e, b\}$. If $b_e = b_f$ for $e \neq f$, then $M|\{b, b_e, e, f\}$ is isomorphic to $U_{2,4}$ and hence $M \backslash e$ is 3-connected; a contradiction. Thus $b_e \neq b_f$ whenever $e \neq f$. Therefore

$$|E(M) - B| \leq |B - b| = r(M) - 1$$

and the result follows. $\square$

We now show that Theorem 1.2 is best-possible. Let $K''_{3,n}$ be the graph that is obtained from $K_{3,n}$ by adding a new edge from one of the degree-n vertices of the latter to each of the other two degree-n vertices. Then equality is attained in the bound in Theorem 1.2 if we take M to be $M(K''_{3,n})$ and B to be the set of edges meeting the vertex of degree $n+2$ in $K''_{3,n}$.

3. The circumference of a 3-connected matroid

For a k-connected graph G,the minimum vertex degree is at least k. When $k \geq 2$, a well-known result of Dirac [2, Theorem 4] implies that the circumference of G is at least $2k$ provided that $|V(G)| \geq 2k$. Moreover, this result is best-possible. Thus a 3-connected graph with at least six vertices has circumference at least six. In this section, we shall prove Theorem 1.5, a generalization of this result to 3-connected matroids having rank at least six.

Let L be a subset of the ground set of a matroid M and suppose that L is the union of a set of circuits of M and $r^*(M|L) = 2$. Then L is what Tutte [12] has called a "line" of M. We shall call L a *Tutte-line* since the word "line" is also commonly used in matroid theory to mean a rank-2 flat. It is not difficult to see that every Tutte-line L of a matroid M has a *canonical partition* $\{L_1, L_2, \ldots, L_k\}$ such that a subset C of L is a circuit of $M|L$ if and only if $C = L - L_i$ for some i in $\{1, 2, \ldots, k\}$. A Tutte-line L is *connected* if $M|L$ is a connected matroid.

In the next proof, we shall make frequent use of the next two lemmas. Both parts of the first of these are elementary consequences of *orthogonality*, the property of a matroid that a circuit and a cocircuit cannot have exactly one common element. The second lemma was proved by Oxley [10, Theorem 2.5].

3.1. Lemma. (i) *If C is a circuit of a 3-connected matroid and T_1 and T_2 are distinct triads of M both of which meet C, then $C \cap T_1 \neq C \cap T_2$.*

(ii) *If L is a Tutte-line of a matroid M and T is a triad of M, then T meets an odd number of sets in the canonical partition of L.*

3.2. Lemma. *Let C be a circuit of a minimally 3-connected matroid M and suppose that $|E(M)| \geq 4$. Then M has at least two distinct triads intersecting C.*

Proof of Theorem 1.5. Let M be a counterexample to the theorem for which $|E(M)|$ is minimal. Clearly M must be minimally 3-connected. Let C be a circuit of M such that $|C| = c(M)$. Then $3 \leq |C| \leq 5$.

Let $\mathcal{D}$ be the set of circuits D of M/C such that D is not a circuit of M and $|D| \geq 2$.

3.3. Lemma. (i) *$|D| = 2$ for every D in $\mathcal{D}$;*

(ii) *$|C| \geq 4$; and*

(iii) *for all D in $\mathcal{D}$, the set $C \cup D$ is a connected Tutte-line of M having canonical partition $\{X_D, Y_D, D\}$ for some X_D and Y_D with $|X_D|, |Y_D|$ in $\{2, 3\}$.*

Proof. Suppose that $D \in \mathcal{D}$. Then $C \cup D$ is a connected Tutte-line L of M. Let $\{L_1, L_2, \ldots, L_k\}$ be the canonical partition of L. As L is connected, $k \geq 3$. Since $(C \cup D) - D$ is a circuit of $M|L$, we may assume, without loss of generality, that $D = L_1$. As $L - L_1 = C$, a maximum-sized circuit of M, we must have that $|L - L_1| \geq |L - L_i|$ for all i. But $|D| \geq 2$. Thus

$$2 \leq |D| = |L_1| \leq |L_i| \tag{4}$$

for all i. As $\cup_{j=2}^{k} L_j = C$ and $|C| \leq 5$, we deduce that $k \leq 3$. Hence $k = 3$. Moreover, $|L_2|, |L_3| \in \{2, 3\}$ and $\min\{|L_2|, |L_3|\} = 2$. Thus, by (4), $|D| = 2$. In addition, $|C| = |L_2| + |L_3| \geq 4$, and (iii) holds. $\square$

3.4. Lemma. $c(M/C) \le 2$.

Proof. Let C' be a circuit of M/C with at least three elements. By Lemma 3.3, $C' \notin \mathcal{D}$. Thus C' is a circuit of M. Observe that $M|(C \cup C') = (M|C) \oplus (M|C')$. Choose an element d of C'. Then M has a circuit C_d that contains d and meets C. Clearly $C_d - C$ is a union of circuits of M/C. Take such a circuit D that contains d. Since D and C' are both circuits of M/C, we cannot have that $D = \{d\}$ otherwise D is a proper subset of C'. Hence $|D| \ge 2$. Lemma 3.3 now implies that $|D| = 2$, say $D = \{d, d'\}$. Moreover, $M|(C \cup D)$ is connected. Since $M|C'$ is also connected, it follows that $M|(C \cup D \cup C')$, which equals $M|(C \cup C' \cup d')$, is connected. Deleting d' from the last matroid produces the disconnected matroid $M|(C \cup C')$ with components $M|C$ and $M|C'$. Thus $M|(C \cup C' \cup d')$ is the series connection, with basepoint d', of $[M|(C \cup C' \cup d')]/C'$ and $[M|(C \cup C' \cup d')]/C$. Now $M|(C \cup C' \cup d')$ has a circuit D' that contains d' and at least two elements of C', otherwise every element of C' is parallel to d, a contradiction to the fact that C' is a circuit of M of size at least three. Then $D' = D'_1 \cup D'_2 \cup d'$ where $D'_1 \cup d'$ and $D'_2 \cup d'$ are circuits of $[M|(C \cup C' \cup d')]/C'$ and $[M|(C \cup C' \cup d')]/C$, respectively. Hence $D' - C = D'_2 \cup d'$ and so $D' - C$ is a circuit of M/C that is not a circuit of M, and $|D' - C| \ge 3$. Thus $D' - C \in \mathcal{D}$ yet Lemma 3.3 fails for it; a contradiction. □

By Lemma 3.4, the connected components of M/C consist of loops and parallel classes. But each parallel class of M/C is a cocircuit of M and therefore has at least three elements. Let these rank-one components of M/C be $H_1, H_2, \dots, H_n$. Then $n \ge 1$ since C does not span M. Therefore, as M is 3-connected and $|E(M) - E(H_i)| \ge |C| \ge 3$, we have that

$$r(E(H_i)) + r(E(M) - E(H_i)) - r(M) \ge 2.$$

Thus, as $E(M) - E(H_i)$ is a hyperplane of M, it follows that

$$r(E(H_i)) \ge 3 \text{ for all } i. \tag{5}$$

3.5. Lemma. *C does not contain a triad of M.*

Proof. Suppose that C contains a triad T of M. Choose a subset A of $E(H_1)$ such that $|A| = 2$. Then $A \in \mathcal{D}$ so, by Lemma 3.3, $C \cup A$ is a connected Tutte-line of M having canonical partition $\{A, X, Y\}$ with $|X|, |Y| \in \{2, 3\}$. Since $X \cup Y = C$, it follows, by Lemma 3.1(ii), that $\{X, Y\} = \{T, C - T\}$. Hence, as $|C| \le 5$, we deduce that $|C - T| \le 2$. Thus $|C - T| = 2$ and $|C| = 5$. Moreover, for all 2-element subsets A of $E(H_1)$, the set $A \cup (C - T)$ is a circuit C_A of M.

As M is minimally 3-connected, Lemma 3.2 implies that M has distinct triads T_1 and T_2 both of which meet C_A. By Lemma 3.1(i), $T_1 \cap C_A \ne T_2 \cap C_A$. Thus, as $|A| = 2$, at least one of T_1 and T_2, say the former, meets $C - T$. Now, by Lemma 3.1(ii), either (i) $T_1 \cap C \subseteq C - T$; or (ii) $T_1 = \{a, t, c\}$ for some a in A, some t in T, and some c in $C - T$. Let A' be a subset of $E(H_1) - a$ such that $|A \cap A'| = 1$. Then $C_{A'}$, which equals $A' \cup (C - T)$, cannot meet T_1 in a single element, so (ii) does not hold. Hence (i) holds.

Let A' be a subset of $E(H_1)$ such that $|A' \cap A| = 1$. By Lemma 3.1(ii), $A' \cap T_1 = \emptyset$, since $\{A', T, C - T\}$ is the canonical partition of $A' \cup C$. Let $C - T = \{x, y\}$. Then $(C_A \cup C_{A'}) - y$, which equals $A \cup A' \cup x$, contains a circuit of M. As $|A \cup A'| = 3$ and $(A \cup A' \cup x) \cap T_1 = \{x\}$, it follows that $A \cup A'$ is a triangle of M. As A' was an arbitrarily chosen subset of $E(H_1)$ for which $|A \cap A'| = 1$, we deduce that A spans $E(H_1)$ in M. Hence $r(E(H_1)) = 2$; a contradiction to (5). □

3.6. Lemma. *If $i \in \{1, 2, \dots, k\}$ and $\{a, b\}$ is a 2-element subset of $E(H_i)$ that is contained in a triad of M, then $E(H_i)$ is a triad of M.*

Proof. Let T be a triad of M that contains $\{a, b\}$ and suppose that $T \cap E(H_i) = \{a, b\}$. By orthogonality, $T \cap C = \emptyset$. Thus $\{a, b\}$ is a union of cocircuits of $M|(C \cup E(H_i))$. But, since $E(H_i)$ is a parallel class of M/C of size at least three, it follows that $r(C \cup E(H_i)) = r(C) + 1$ and $E(H_i)$ is a cocircuit of $M|(C \cup E(H_i))$. This is a contradiction since $\{a, b\}$ is a proper subset of $E(H_i)$. We conclude that $T \subseteq E(H_i)$. Since both T and $E(H_i)$ are cocircuits of M, equality must hold here. □

3.7. Lemma. *For all i, the matroid $M|(C \cup E(H_i))$ has at most one 2-cocircuit contained in C.*

Proof. Suppose that $M|(C \cup E(H_i))$ has at least two 2-cocircuits W_1 and W_2 contained in C. We now distinguish the following two cases: (i) $|W_1 \cap W_2| = 1$; and (ii) $|W_1 \cap W_2| = 0$. In each case, we shall show that every 3-element subset of $E(H_i)$ is a triangle so $r(E(H_i)) = 2$; a contradiction to (5).

Assume that (i) holds. Then $W_1 \cup W_2$ is contained in a series class of $M|(C \cup E(H_i))$. Let $\{a, b, c\}$ be an arbitrary 3-element subset of $E(H_i)$. Then Lemma 3.3 implies that, for each A in $\{\{a, b\}, \{b, c\}\}$, the set $C \cup A$ is a connected Tutte-line of M having canonical partition $\{A, X_A, Y_A\}$ where each of X_A and Y_A has either two or three elements. Since both $A \cup X_A$ and $A \cup Y_A$ are circuits of M, we must have that $W_1 \cup W_2$ is contained in and therefore equals X_A or Y_A. Thus both $W_1 \cup W_2 \cup \{a, b\}$ and $W_1 \cup W_2 \cup \{b, c\}$ are circuits of M. Hence if $w \in W_1$, then $(W_1 \cup W_2 \cup \{a, b, c\}) - w$ contains a circuit C_w of M. As $W_1 \cup W_2$ is contained in a series class of $M|(C \cup E(H_i))$ and $w \notin C_w$, it follows that C_w avoids $W_1 \cup W_2$. Hence C_w is contained in and therefore equals $\{a, b, c\}$. It follows that $r(E(H_i)) = 2$; a contradiction to (5).

We may now assume that (ii) holds. Let $\{a, b, c\}$ be an arbitrary 3-element subset of $E(H_i)$. Then, for each 2-element subset A of $\{a, b, c\}$, Lemma 3.3 implies that $C \cup A$ is a connected Tutte-line. Moreover, for the canonical partition $\{A, X_A, Y_A\}$ of $C \cup A$, we must have that each of W_1 and W_2 is contained in X_A or Y_A. The fact that each of X_A and Y_A has two or three elements, but their union has at most five elements implies that $A \cup W_1$ or $A \cup W_2$ is a circuit of M. Hence there are distinct 2-element subsets A and A' of $\{a, b, c\}$ such that, for some j in $\{1, 2\}$, both $A \cup W_j$ and $A' \cup W_j$ are circuits of M. Thus, if $w \in W_j$, then M has a circuit C_1 contained in $(A \cup A' \cup W_j) - w$. Since W_j is a 2-cocircuit of $M|(C \cup E(H_i))$, it follows that C_1 is contained in and hence equals $A \cup A'$. Therefore, as in case (i), we deduce that every 3-element subset of $E(H_i)$ is a triangle, so $r(E(H_i)) = 2$; a contradiction to (5). □

3.8. Lemma. *Suppose that $i \in \{1, 2, \dots, k\}$ and that $\{a, b, c\}$ is a 3-element subset of $E(H_i)$. Then $\{a, b\}$ is contained in a 4-circuit D of M where $|D \cap C| = 2$. Moreover, either*

(i) *M has a triad that meets both $\{a, b\}$ and $D \cap C$; or*
(ii) *$E(H_i)$ is a triad of M and there is a triad of M that contains c and meets C.*

In both cases, M has a triad that meets both $E(H_i)$ and C.

Proof. The fact that D exists follows immediately from Lemma 3.3(iii) since $|C| \leq 5$. Let $D \cap C = \{\alpha, \beta\}$. Suppose that (i) does not occur. By Lemma 3.2, there are triads T and T' of M that meet D. By Lemma 3.1(i), $T \cap D \neq T' \cap D$. The canonical partition of the connected Tutte-line $\{a,b\}\cup C$ must be $\{\{a,b\},\{\alpha,\beta\},C-\{\alpha,\beta\}\}$. Thus orthogonality and Lemma 3.1(ii) imply that each of $|T\cap(C\cup\{a,b\})|$ and $|T'\cap(C\cup\{a,b\})|$ is 2. Hence we may assume, without loss of generality, that $T\cap(C\cup\{a,b\})=\{a,b\}$ and $T'\cap(C\cup\{a,b\})=\{\alpha,\beta\}$. Therefore, by Lemma 3.6, $E(H_i)=\{a,b,c\}$ and $E(H_i)$ is a triad of M. We may suppose that the element of $T'-(C\cup\{a,b\})$ is not in $C\cup E(H_i)$ otherwise this element is c and (ii) follows.

Applying Lemma 3.2 again, this time to the circuit C, we deduce that M has a triad T'' that meets C and is different from T'. By Lemma 3.5, $|T''\cap C|=2$. We show next that $T''\cap E(H_i)$ is empty. If not, then $c\in T''$, or, by Lemma 3.1(ii), T'' meets both $\{a,b\}$ and $D\cap C$. But, in both these cases, the lemma holds. Thus $T''\cap E(H_i)$ is indeed empty. Hence $T''\cap C$ is a 2-cocircuit of $M|(C\cup E(H_i))$. But $T'\cap C$ is also a 2-cocircuit of $M|(C\cup E(H_i))$ and Lemma 3.1(i) implies that $T'\cap C\neq T''\cap C$. Hence $M|(C\cup E(H_i))$ has two 2-cocircuits contained in C. This contradiction to Lemma 3.7 completes the proof of Lemma 3.8. □

3.9. Lemma. *M has exactly two rank-one components, H_i. Moreover, $|C|=5$ and $r(M)=6$.*

Proof. Suppose that the number n of rank-one components of M/C is at least three. Then, by Lemma 3.8, for each i in $\{1,2,\dots,n\}$, there is a triad T_i such that both $T_i\cap C$ and $T_i\cap E(H_i)$ are non-empty. By Lemma 3.1(ii), if i and j are distinct members of $\{1,2,\dots,n\}$, then $T_i\cap C\neq T_j\cap C$. Thus $M|(C\cup E(H_1))$ has both $T_2\cap C$ and $T_3\cap C$ as 2-cocircuits contained in C, a contradiction to Lemma 3.7. We conclude that $n\le 2$. Thus

$$6\le r(M)=|C|-1+n\le |C|+1\le 6,$$

so $r(M)=6$. Moreover, $n=2$ and $|C|=5$. □

We now work towards obtaining a final contradiction that will complete the proof of Theorem 1.5. By Lemma 3.8, for each i in $\{1,2\}$, there is a triad T_i that meets both $E(H_i)$ and C. If, for a fixed i, there are two such triads $T_{i,1}$ and $T_{i,2}$, then, for $j\neq i$, the sets $T_{i,1}\cap C$ and $T_{i,2}\cap C$ are 2-cocircuits of $M|(C\cup E(H_j))$. Thus, by Lemma 3.7, $T_{i,1}\cap C=T_{i,2}\cap C$, a contradiction to Lemma 3.1(ii). Hence T_i is unique.

Suppose that, for some i in $\{1,2\}$, the set $E(H_i)$ is not a triad of M. Then Lemma 3.8 implies that, for each 2-element subset A of $E(H_i)$, there is a 4-element circuit, D_A, that contains A and meets C in exactly two elements. Thus, as (ii) of Lemma 3.8 does not hold, (i) of that lemma implies that there are at least two triads of M that meet C and $E(H_i)$. This contradiction to the uniqueness of T_i implies that both $E(H_1)$ and $E(H_2)$ are triads of M.

Let $E(H_1)=\{a,b,c\}$ and $E(H_2)=\{a',b',c'\}$. Without loss of generality, we may suppose that $T_1\cap E(H_1)=\{c\}$ and $T_2\cap E(H_2)=\{c'\}$. Then, for some $\{\alpha,\beta\}\subseteq C$, there is a circuit $\{\alpha,\beta,a,b\}$ of M. By orthogonality, either

(i) $T_1=\{\alpha,\beta,c\}$; or
(ii) $T_1\cap\{\alpha,\beta,a,b\}=\emptyset$.

In both cases, we shall show that

(iii) $T_1\cap T_2=\emptyset$; and
(iv) there are distinct elements μ and ν of T_1-c so that $\{a,c,\varepsilon,\mu\}$ and $\{b,c,\varepsilon,\nu\}$ are circuits of M where ε is the element of $C-(T_1\cup T_2)$.

Suppose that (ii) occurs. Then, by Lemma 3.2, M has a triad T that meets $\{\alpha,\beta,a,b\}$ and is different from $E(H_1)$. Since $E(H_1)\cap\{\alpha,\beta,a,b\}=\{a,b\}$, it follows, by Lemma 3.1(ii), that $T\cap\{\alpha,\beta,a,b\}\neq\{a,b\}$. Thus either T meets both $E(H_1)$ and C, or $T\cap\{\alpha,\beta,a,b\}=\{\alpha,\beta\}$. Since T_1 is the only triad meeting both $E(H_1)$ and C, the first case implies that $T_1=T$; a contradiction since T_1 avoids $\{\alpha,\beta,a,b\}$ by (ii). We conclude that $T\cap\{\alpha,\beta,a,b\}=\{\alpha,\beta\}$. If T avoids $E(H_2)$, then $T_1\cap C$ and $T\cap C$ are distinct 2-cocircuits of $M|(C\cup E(H_2))$ contained in C; a contradiction to Lemma 3.7. Thus T meets $E(H_2)$. Since T also meets C, it follows that $T=T_2$ and hence that $T_2=\{\alpha,\beta,c'\}$. Hence $T_1\cap T_2=\emptyset$, that is, (iii) holds. We may now assume that $T_1=\{\gamma,\delta,c\}$ where $C=\{\alpha,\beta,\gamma,\delta,\varepsilon\}$. By Lemma 3.3 and the fact that $|C|=5$, the connected Tutte-line $C\cup\{a,c\}$ contains a 5-element circuit C_1 of M that contains $\{a,c\}$. By applying Lemma 3.1(ii) to T_1, we deduce that $|C_1\cap\{\gamma,\delta\}|=1$, say $\gamma\in C_1$. As $T_2=\{\alpha,\beta,c'\}$, it follows that $\{\alpha,\beta\}$ is a series class of $M|(C\cup\{a,c\})$ and hence that $C_1=\{\alpha,\beta,\gamma,a,c\}$. Thus $\{a,c,\delta,\varepsilon\}$ is a circuit of M.

Now use the connected Tutte-line $C\cup\{b,c\}$ to give that M has a 5-element circuit C_2 that contains $\{b,c\}$. Then, arguing as for C_1, we deduce that $\{b,c,\delta,\varepsilon\}$ or $\{b,c,\gamma,\varepsilon\}$ is a circuit of M. The first possibility is excluded since it implies that $\{a,b,\delta,\varepsilon\}$ contains a circuit of M. This contradicts the fact that $\{\{a,b\},\{\alpha,\beta\},\{\gamma,\delta,\varepsilon\}\}$ is the canonical partition of $C\cup\{a,b\}$. We conclude that if (ii) occurs, then (iii) and (iv) hold.

Now suppose that (i) holds. Then, by Lemma 3.1(i) and (ii), $T_1\cap T_2=\emptyset$, that is, (iii) holds. Thus we may assume that $T_2=\{\gamma,\delta,c'\}$. Let C_3 be a 5-element circuit of $M|(C\cup\{a,c\})$ that contains $\{a,c\}$. Then, by applying Lemma 3.1(ii) to the triad $\{\alpha,\beta,c\}$, we have that $|C_3\cap\{\alpha,\beta\}|=1$, say $\alpha\in C_3$. As $T_2=\{\gamma,\delta,c'\}$, it follows that $\{\gamma,\delta\}$ is a cocircuit of $M|(C\cup\{a,c\})$. Thus the canonical partition of $C\cup\{a,c\}$ is $\{\{a,c\},\{\alpha,\gamma,\delta\},\{\beta,\varepsilon\}\}$. Hence $\{a,c,\beta,\varepsilon\}$ is a circuit of M. Arguing similarly using a 5-element circuit C_4 of $M|(C\cup\{b,c\})$ that contains $\{b,c\}$, we deduce that $\{b,c,\beta,\varepsilon\}$ or $\{b,c,\alpha,\varepsilon\}$ is a circuit of M. The first possibility cannot occur because it implies that $\{a,b,\beta,\varepsilon\}$ contains a circuit of M which contradicts the fact that $\{\{a,b\},\{\alpha,\beta\},\{\gamma,\delta,\varepsilon\}\}$ is the canonical partition of $C\cup\{a,b\}$. We conclude that $\{b,c,\alpha,\varepsilon\}$ is a circuit of M and hence that, when (i) occurs, both (iii) and (iv) hold.

Since $\{\{a,b\},\{\alpha,\beta\},\{\gamma,\delta,\varepsilon\}\}$ is the canonical partition of $C\cup\{a,b\}$ and $T_1\cap T_2=\emptyset$, Lemma 3.1(ii) implies that either $T_1=\{\alpha,\beta,c\}$ and $T_2=\{\gamma,\delta,c'\}$, or $T_1=\{\gamma,\delta,c\}$ and $T_2=\{\alpha,\beta,c'\}$. Using Lemma 3.1(ii) again, it follows that, in each case, either $\{\{a',b'\},\{\alpha,\beta,\varepsilon\},\{\gamma,\delta\}\}$ or $\{\{a',b'\},\{\alpha,\beta\},\{\gamma,\delta,\varepsilon\}\}$ is the canonical partition of $C\cup\{a',b'\}$. Thus $\{a',b',\gamma,\delta\}$ or $\{a',b',\alpha,\beta\}$ is a circuit of M. Using this circuit in place of $\{a,b,\alpha,\beta\}$, and $E(H_2)$ and T_2 in place of $E(H_1)$ and T_1, we may now argue as in (i) and (ii) above to deduce that there are distinct elements μ' and ν' of T_2-c' such that $\{a',c',\varepsilon,\mu'\}$ and $\{b',c',\varepsilon,\nu'\}$ are circuits of M.

By elimination, $(\{a,c,\varepsilon,\mu\}\cup\{a',c',\varepsilon,\mu'\})-\varepsilon$ contains a circuit C_5 of M. The triads $\{a,b,c\}$ and T_1 and orthogonality imply that either $C_5\cap\{a,c,\mu\}=\emptyset$, or $\{a,c,\mu\}\subseteq C_5$. Similarly, the triads $\{a',b',c'\}$ and T_2 imply that $C_5\cap\{a',c',\mu'\}=\emptyset$, or $\{a',c',\mu'\}\subseteq C_5$. But $|C_5|\leq 5$, so C_5 is either $\{a,c,\mu\}$ or $\{a',c',\mu'\}$. However, the connected Tutte-lines $C\cup\{a,c\}$ and $C\cup\{a',c'\}$ imply that both $\{a,c,\mu\}$ and $\{a',c',\mu'\}$ are properly contained in circuits of M. This contradiction completes the proof of Theorem 1.5. □

To close this section, we shall present an example that shows that the lower bound on the rank in the hypothesis of Theorem 1.5 is best-possible. Let M be the tipless binary 5-spike, that is, M is the matroid that is represented over $GF(2)$ by the matrix $[I_5|J_5 - I_5]$ where J_5 is the 5×5 matrix of all ones. It is not difficult to show (see, for example, [11, p.321]) that M has circumference 5.

4. Removing circuits from matroids

In this section, we shall prove Theorems 1.6 and 1.7.

Proof of Theorem 1.6. By Lemma 2.1, there is a 3-connected matroid N having the same rank and the same circumference as M such that $N = M|E(N)$ and

$$|E(N)| \leq 2r(M) + \max\{0, r(M) - c(M)\}.$$

Now

$$\begin{aligned} |E(M)| - |E(N)| &\geq [3r(M) + 1 + \max\{0, r(M) - c(M)\}] \\ &\qquad -[2r(M) + \max\{0, r(M) - c(M)\}] \\ &= r(M) + 1. \end{aligned}$$

Thus $E(M) - E(N)$ must be a dependent set of M and so contains a circuit C of M avoiding $E(N)$. Observe that $M\backslash C$ is a 3-connected matroid, since it has N as a minor and $E(N)$ spans M. $\square$

On combining the last theorem with Theorem 1.5, we immediately obtain the following result.

4.1. Corollary. *If M is a 3-connected matroid such that $r(M) \geq 6$ and*

$$|E(M)| \geq 4r(M) - 5,$$

then M has a circuit C such that $M\backslash C$ is 3-connected.

The last corollary is best-possible as the next example shows. Let M be the matroid that is obtained as follows. Begin with a 3-point line $\{a, b, c\}$ and take the generalized parallel connection of n copies, $N_1, N_2, \dots, N_n$, of $M(K_4)$ across $\{a, b, c\}$. Each N_i has a unique 3-point line that meets $\{a, b, c\}$ at a. Freely add a point p_i on each such line. Then M is obtained by deleting a from the resulting matroid. Certainly M is 3-connected and has rank $2 + n$. Moreover, for each i, the 4-element set $(E(N_i) \cup p_i) - \{a, b, c\}$ is a cocircuit of M containing a triangle. Using this fact and orthogonality, it is not difficult to see that M has no circuit whose deletion leaves a 3-connected matroid. But

$$|E(M)| = 4n + 2 = 4(n + 2) - 6 = 4r(M) - 6,$$

so the bound in the last corollary cannot be improved.

The proof of Theorem 1.7 is obtained by making slight modifications to the proof of Theorem 1.6 so that Theorem 1.2 rather than Lemma 2.1 can be used. We omit the straightforward details.

To obtain an example showing that Theorem 1.7 is best-possible, we modify the previous example by freely adding two new points q_1 and q_2 on the line $\{b, c\}$. Let the resulting matroid be M'. Then $|E(M')| = 4n + 4$ and $r(M') = n + 2$, so $|E(M')| = 4r(M') - 4$. For each i, let b_i be the coloop of $M'|[(E(N_i) \cup p_i) - \{a, b, c\}]$ and let $B = \{b, c, b_1, b_2, \dots, b_n\}$. Then B is a basis for M' and, arguing as for M, it is not difficult to see that M' has no circuit C that avoids B such that $M'\backslash C$ is 3-connected.

Acknowledgements The first author was partially supported by CNPq, CAPES, FINEP and PRONEX 107/97. The second author was partially supported by the National Security Agency.

References

[1] Cunningham, W.H., *A combinatorial decomposition theory*, Ph. D. thesis, University of Waterloo, 1973.
[2] Dirac, G.A., Some theorems on abstract graphs, *Proc. London Math. Soc. (3)* **2** (1952), 69–81.
[3] Halin, R., Zur Theorie der n-fach zusammenhängenden Graphen, *Abh. Math. Sem. Univ. Hamburg* **33** (1969), 133–164.
[4] Jackson, B., Removable cycles in 2-connected graphs of minimum degree at least four, *J. London Math. Soc. (2)* **21** (1980), 385–392.
[5] Lemos, M. and Oxley, J., On removable circuits in graphs and matroids, *J. Graph Theory*, to appear.
[6] Lemos, M. and Oxley, J., On packing minors into connected matroids, *Discrete Math.*, to appear.
[7] Lemos, M. and Oxley, J., On the 3-connected matroids that are minimal having a fixed spanning restriction, submitted.
[8] Lemos, M., Oxley, J., and Reid, T.J., On the 3-connected matroids that are minimal having a fixed restriction, submitted.
[9] Mader, W., Kreuzungfreie a, b-Wege in endlichen Graphen, *Abh. Math. Sem. Univ. Hamburg* **42** (1974), 187–204.
[10] Oxley, J.G., On matroid connectivity, *Quart. J. Math. Oxford Ser. (2)* **32** (1981), 193–208.
[11] Oxley, J.G., *Matroid theory,* Oxford University Press, New York, 1992.
[12] Tutte, W.T., Lectures on matroids, *J. Res. Nat. Bur. Standards Sect. B* **69B** (1965), 1–47.

Departamento de Matemática, Universidade Federal de Pernambuco, Recife, Pernambuco 50740-540, Brazil
E-mail address: manoel@dmat.ufpe.br

Department of Mathematics, Louisiana State University, Baton Rouge, Louisiana 70803–4918, USA
E-mail address: oxley@math.lsu.edu