PACKING AND COVERING TRIANGLES IN TRIPARTITE GRAPHS

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Abstract. It is shown that if G is a tripartite graph such that the maximum size of a set of pairwise edge-disjoint triangles is $\nu(G)$, then there is a set C of edges of G of size at most $(2-\varepsilon)\nu(G)$ such that $E(T)\cap C\neq\emptyset$ for every triangle T of G, where $\varepsilon>0.044$. This improves the previous bound of $(7/3)\nu(G)$ due to Tuza [**6**].

§0. Introduction

In this note we are concerned with a packing and covering problem for triangles in graphs. Let a graph G be given. We say that a family $\mathcal{F}$ of triangles in G is *independent* if the elements of $\mathcal{F}$ are pairwise edge-disjoint. Moreover, we say that a set $C\subset E(G)$ of edges of G is a *transversal of the set of triangles of* G, or simply a *transversal* for G, if every triangle of G contains at least one element of C. We denote by $\nu(G)$ the maximum cardinality of an independent family of triangles in G, and by $\tau(G)$ the minimum cardinality of a transversal for G. It is clear that for every graph G we have $\nu(G)\leq\tau(G)\leq 3\nu(G)$. The graphs $G=K^4$ and K^5 show that we may have $\tau(G)$ as large as $2\nu(G)$. In this note, we are concerned with an elegant conjecture of Tuza, first raised fifteen years ago [**4**], which states that $\tau(G)\leq 2\nu(G)$ for every graph G.

Some partial results on this conjecture have been obtained for certain special classes of graphs. In [**6**], Tuza shows that the conjecture is true for planar graphs, chordal graphs, and graphs with n vertices and at least $7n^2/16$ edges. More recently, Krivelevich [**3**] has shown that graphs G without homeomorphic copies of $K^{3,3}$ satisfy the conjecture, improving on Tuza's result that the conjecture holds for planar graphs. Two fractional versions of the conjecture are considered by Krivelevich [**3**] as well. Tuza [**6**] also shows that for tripartite graphs G one has $\tau(G)\leq 7\nu(G)/3$. In this note we prove that $\tau(G)\leq(2-\varepsilon)\nu(G)$ for tripartite graphs G, where $\varepsilon>0.044$. We give a short, self-contained proof of the inequality $\tau(G)\leq 2\nu(G)$ (see Lemma 1); as we shall see, the difficulty will lie in getting the ε improvement. It is interesting to note that it has not even been proved that there is a constant $c<3$ such that $\tau(G)<c\nu(G)$ for *every* graph G.

Our result is also a special case of a conjecture of Ryser on packing and covering in tripartite hypergraphs. A tripartite hypergraph $\mathcal{H}$ consists of three disjoint sets of vertices X_1, X_2, and X_3, and a set $\mathcal{E}$ of hyperedges E, where each hyperedge is

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a set of three vertices, one from each X_i. Ryser conjectured that $\tau(\mathcal{H}) \leq 2\nu(\mathcal{H})$ for every tripartite hypergraph $\mathcal{H}$, where here $\nu(\mathcal{H})$ denotes the maximum size of a set of pairwise disjoint hyperedges of $\mathcal{H}$, and $\tau(\mathcal{H})$ is the minimum size of a set of vertices C such that $E \cap C \neq \emptyset$ for all $E \in \mathcal{E}$. The best upper bound for this problem that is currently known **[2]** is $\tau(\mathcal{H}) \leq 5\nu(\mathcal{H})/2$. For more details on Ryser's conjecture, see Conjecture 4.29 in Füredi **[1]**, and Tuza **[5]**. Note that if G is a tripartite graph then its edge set can be partitioned into 3 classes such that every triangle in G contains exactly one edge from each class. Therefore $\mathcal{T}(G)$, the set of triangles in the graph G, corresponds in an obvious way to the set of hyperedges of a tripartite hypergraph $\mathcal{H} = \mathcal{H}(G)$, and in our notation $\tau(G) = \tau(\mathcal{H})$ and $\nu(G) = \nu(\mathcal{H})$. Thus our main result of this note shows that Ryser's conjecture is true for hypergraphs that are derived from tripartite graphs in this way.

In §1 of this note we give two lemmas needed in the proof of our main result, Theorem 4. In §2 we give the proof of Theorem 4, and also give examples of tripartite graphs G with $\tau(G) \geq 5\nu(G)/4$.

§1. Two Lemmas

We begin with some definitions and notation. Let G be a tripartite graph with vertex sets V_1, V_2, and V_3, and denote by E_σ the set of edges with one endpoint in $V_{\sigma+1}$ and the other in $V_{\sigma+2}$ (here addition is taken modulo 3). We sometimes say that the vertices in V_σ and the edges in E_σ are coloured with colour σ. For a path, triangle or set of triangles S in G, we shall write $E(S)$ for the set of all the edges appearing in S, and $E_\sigma(S)$ for the subset of $E(S)$ consisting of the edges in E_σ. In the sequel, if the vertices a, b, and c induce a triangle $T = G[\{a, b, c\}]$ in a graph G, we shall write abc for this triangle T. Recall that we denote the family of triangles of a graph G by $\mathcal{T}(G)$.

Lemma 1. *Let $\sigma \in \{1, 2, 3\}$ be fixed, and let $\mathcal{B}$ be an independent set of triangles in G. Then one of the following assertions holds.*

(1.1) *There exists an independent set of triangles $\mathcal{B}'$ in G with $|\mathcal{B}'| = |\mathcal{B}| + 1$ such that $E_{\sigma'}(\mathcal{B}) \subset E_{\sigma'}(\mathcal{B}')$ for both $\sigma' \neq \sigma$.*

(1.2) *We have $\tau(G) \leq 2|\mathcal{B}|$.*

Proof. We shall describe how to construct either $\mathcal{B}'$ or a transversal C for G with $|C| \leq 2|\mathcal{B}|$. Observe that we may clearly assume that $\mathcal{B}$ is a maximal independent family of triangles. In other words, we may assume that every triangle in G contains at least one edge of $E(\mathcal{B})$.

For each vertex $v \in V_\sigma$, let $\mathcal{B}_v$ denote the subset of $\mathcal{B}$ consisting of triangles containing v. We define for each $v \in V_\sigma$ a bipartite graph H_v with vertex set $V_{\sigma+1} \cup V_{\sigma+2}$ as follows. We let xy be an edge of H_v if vxy is a triangle in G and xy is not an edge of T for any $T \in \mathcal{B} \setminus \mathcal{B}_v$. We denote by $\nu(H_v)$ the maximum size of a matching in H_v. Note that $\mathcal{B}_v$ corresponds to a matching in H_v, which we shall denote by M_v. Note also that since $\mathcal{B}$ is maximal, we have that M_v is a maximal matching, *i.e.*, every edge of H_v not in M_v is adjacent to an edge of M_v.

Case 1. For some $v \in V_\sigma$, we have $|M_v| < \nu(H_v)$.

In this case, pick an augmenting path $P = x_1y_1x_2y_2 \ldots x_ky_k$ in H_v, so that $x_jy_j \notin M_v$ $(1 \leq j \leq k)$, $y_jx_{j+1} \in M_v$ $(1 \leq j < k)$, and neither x_1 nor y_k is covered by M_v.

Then the set of triangles

$$\mathcal{B}' = \Big(\mathcal{B} \setminus \{vyx : yx \in E(P) \cap M_v\}\Big) \cup \{vxy : xy \in E(P) \setminus M_v\}$$

is such that $|\mathcal{B}'| = |\mathcal{B}| + 1$. We now check that the set $\mathcal{B}'$ is independent. It is clear that those elements of $\mathcal{B}'$ containing v are independent, as $(E(P)\setminus M_v)\cup(M_v\setminus E(P))$ is a matching in H_v. Also, if $T \in \mathcal{B}'$ does not contain v then it can intersect a triangle containing v only in the edge of colour σ. But by construction, no edge of colour σ of any triangle in $\mathcal{B} \setminus \mathcal{B}_v$ appears in H_v. Finally we note that $E_{\sigma+1}(\mathcal{B} \setminus \mathcal{B}') \cup E_{\sigma+2}(\mathcal{B} \setminus \mathcal{B}') = \{vu : u \in V(P) \setminus \{x_1, y_k\}\} \subset E_{\sigma+1}(\mathcal{B}') \cup E_{\sigma+2}(\mathcal{B}')$. Therefore $\mathcal{B}'$ satisfies the conditions of (1.1), and so this case is finished.

Case 2. For all $v \in V_\sigma$, we have $|M_v| = \nu(H_v)$.

In this case we describe a transversal C of $\mathcal{T}(G)$. For each $v \in V_\sigma$ let C_v denote a vertex-cover of the edges of H_v of minimum size, so $|C_v| = \nu(H_v)$ by the König–Egerváry theorem. We let

$$C = E_\sigma(\mathcal{B}) \cup \bigcup_{v \in V_\sigma} \{vz : z \in C_v\}.$$

Then C is a transversal, since if $T \in \mathcal{T}(G)$ is such that $E(T) \cap E_\sigma(\mathcal{B}) = \emptyset$, then T is represented as an edge e of H_v where v is the vertex of T of colour σ, and so $vz \in E(T) \cap C$ where $z \in C_v$ is a vertex covering e in H_v. To estimate $|C|$, simply notice that $\sum_{v \in V_\sigma} |C_v| = \sum_{v \in V_\sigma} \nu(H_v) = |\mathcal{B}|$. Therefore $|C| = 2|\mathcal{B}|$ and hence (1.2) is proved. $\square$

Note that Lemma 1 immediately implies that $\tau(G) \le 2\nu(G)$ for any tripartite graph G. Our aim now is to improve on Lemma 1 taking into account certain special edges present in the family $\mathcal{B}$.

An edge $e \in E(\mathcal{B})$ appearing in an independent set of triangles $\mathcal{B}$ is said to be *essential* if there exists a triangle T of G such that $E(T) \cap E(\mathcal{B}) = \{e\}$, and we say that such a triangle T is *associated with* e. We denote by $\bar{E}_\sigma(\mathcal{B})$ the set of edges in E_σ that are essential in $\mathcal{B}$, and we let $\nu_\sigma(\mathcal{B}) = |\bar{E}_\sigma(\mathcal{B})|$. A refinement of Lemma 1 is as follows.

Lemma 2. *Let $\sigma \in \{1,2,3\}$ be fixed, and let $\mathcal{B}$ be an independent set of triangles in G. Then one of the following assertions holds.*

(2.1) *There exists an independent set of triangles $\mathcal{B}'$ in G with $|\mathcal{B}'| = |\mathcal{B}| + 1$ such that for both $\sigma' \neq \sigma$, we have $|E_{\sigma'}(\mathcal{B}) \cap E_{\sigma'}(\mathcal{B}')| \ge |\mathcal{B}| - 1$.*

(2.2) *We have $\tau(G) \le 2|\mathcal{B}| - \nu_\sigma(\mathcal{B})$.*

Proof. For each $v \in V_\sigma$ we define $\mathcal{B}_v$, H_v, and M_v as in the proof of Lemma 1, and we assume as before that $\mathcal{B}$ is maximal, and hence that each M_v is a maximal matching in H_v.

A vertex $x \in H_v$ is called *v-essential of type 1* if there is an alternating path $P(x) = y_0x_1y_1x_2y_2\dots x_k$ with $x_1 = x$, $k \ge 1$, $x_jy_j \notin M_v$ $(1 \le j \le k)$ and $y_jx_{j+1} \in M_v$ $(0 \le j < k)$, and vx_k is an essential edge for $\mathcal{B}$. The vertex x is said to be v-essential of *type 2* if the path $P(x)$ as above exists, $k \ge 2$, and $y_{k-1}x_k$ rather than vx_k is an essential edge. A *v-essential* vertex is one that is v-essential of type 1 or type 2. Note that if x is v-essential then x_i is also v-essential for

$i = 1, \ldots, k-1$, and also so is x_k if x is of type 1. We denote by $\bar{V}(H_v)$ the set of v-essential vertices of H_v.

To prove Lemma 2, we assume that there does not exist an independent set $\mathcal{B}'$ of triangles as in (2.1) of our lemma, and prove the following assertion.

(*) *For every vertex $v \in V_\sigma$, we have*

(*i*) *each edge of M_v is incident with at most one v-essential vertex,*
(*ii*) *the set $C = E_\sigma(\mathcal{B}) \cup C_2 \cup C_3$ is a transversal for G, where*

$$C_2 = \{vx : x \in \bar{V}(H_v)\},$$
$$C_3 = \{vy : y \in V_{\sigma+1} \text{ and } wy \in M_v \text{ for some } w \text{ such that } wy \notin \bar{E}_\sigma(\mathcal{B}), \text{ and } w,\, y \notin \bar{V}(H_v)\}.$$

Moreover, $|C| \le 2|\mathcal{B}| - \nu_\sigma(\mathcal{B})$.

We now proceed to prove assertion (*). Let $v \in V_\sigma$ be fixed.

Proof of (*i*). Suppose on the contrary that $xz \in M_v$ is such that both x and z are v-essential. Let $P(x) = zxy_1x_2\ldots x_k$ and $P(z) = xzw_1z_2\ldots z_l$ be the alternating paths associated with x and z. We may assume that we have chosen the triple $(xz, P(x), P(z))$ so as to minimise the sum $k+l$. We shall need the following technical lemma, the proof of which is postponed to the end of this section.

Lemma 3. *The paths $P(x)$ and $P(z)$ are vertex-disjoint, except for x and z.*

Assuming the result of Lemma 3, we now proceed to derive a contradiction to the assumption that $\mathcal{B}'$ as in (1.1) of the lemma does not exist. We first note that the following statement holds.

(**) *The family*

$$\mathcal{B}'' = \Big(\mathcal{B} \setminus \{vab : ab \in E(P(x) \cup P(z)) \cap M_v\}\Big) \cup \{vab : ab \in E(P(x) \cup P(z)) \setminus M_v\}$$

is an independent set of triangles of size $|\mathcal{B}| - 1$, and

$$E_{\sigma+1}(\mathcal{B} \setminus \mathcal{B}'') \cup E_{\sigma+2}(\mathcal{B} \setminus \mathcal{B}'') \subset E_{\sigma+1}(\mathcal{B}'') \cup E_{\sigma+2}(\mathcal{B}'') \cup \{vx_k, vz_l\}.$$

Indeed, since $\big[(E(P(x)) \cup E(P(z))) \setminus M_v\big] \cup \big[M_v \setminus (E(P(x)) \cup E(P(z)))\big]$ is a matching in H_v, the set $\mathcal{B}''_v$ of triangles in $\mathcal{B}''$ containing v are all edge-disjoint. Also, a triangle in $\mathcal{B}''_v$ can intersect a triangle in $\mathcal{B}'' \setminus \mathcal{B}''_v = \mathcal{B} \setminus \mathcal{B}_v$ only in the edge of colour σ, but this is not possible since the edges in $E_\sigma(\mathcal{B} \setminus \mathcal{B}_v)$ are not present in H_v by construction. Therefore $\mathcal{B}''$ is independent, and also clearly has size $|\mathcal{B}| - 1$. The edge condition in (**) follows immediately from the definition.

Now to obtain the desired contradiction, we shall show that it is possible to add two more triangles to $\mathcal{B}''$ to form an independent set of triangles satisfying (2.1) of the lemma. Note that the edge condition in (2.1) will then follow directly from the edge condition in (**). We shall consider three different cases, according to the type of the vertices x and z.

Case 1. The vertices x and z are both of type 1.

Let vx_kc and vz_ld be triangles associated with the v-essential edges vx_k and vz_l respectively. Let

$$\mathcal{B}' = \mathcal{B}'' \cup \{vx_kc, vz_ld\}.$$

Note that vx_kc and vz_ld are also edge-disjoint from each other and the rest of $\mathcal{B}'_v = \{vxy \in \mathcal{B}'\}$, since c and d are not incident with any edge of M_v as vx_k and vz_l are essential for $\mathcal{B}$, and hence in particular are not equal to any vertex of $P(x) \cup P(z)$. Also $c \neq d$ since they are in different vertex classes. Therefore $\mathcal{B}'$ is a set of pairwise edge-disjoint triangles satisfying (2.1).

Case 2. The vertex x is of type 1 and z is of type 2.

Let vx_kd and $cw_{l-1}z_l$ be triangles associated with the v-essential edges vx_k and $w_{l-1}z_l$ respectively, where $v \neq c \in V_\sigma$. Let

$$\mathcal{B}' = \mathcal{B}'' \cup \{vx_kd, cw_{l-1}z_l\}.$$

The proof that vx_kd is edge-disjoint from $\mathcal{B}' \setminus \{vx_kd, cw_{l-1}z_l\}$ is as in Case 1, and the fact that $v \neq c$ and $x_k \neq w_{l-1}$ is enough to prove that $cw_{l-1}z_l$ is edge-disjoint from vx_kd. Also, $cw_{l-1}z_l$ is edge-disjoint from all of $\mathcal{B}' \cap \mathcal{B}$ since $w_{l-1}z_l$ is essential for $\mathcal{B}$ and $w_{l-1}z_l \in (E(P(x)) \cup E(P(z))) \cap M_v$, and from $\mathcal{B}' \setminus \mathcal{B}$ since $w_{l-1}z_l \notin (E(P(x)) \cup E(P(z))) \setminus M_v$ and $c \neq v$. Therefore $\mathcal{B}'$ is an independent set of triangles that satisfies (2.1).

Case 3. The vertices x and z are both of type 2.

Let $y_{k-1}x_kd$ and $cw_{l-1}z_l$ be triangles associated with the v-essential edges $y_{k-1}x_k$ and $w_{l-1}z_l$ respectively, where c, $d \in V_\sigma$ and $v \neq c$, d. This time let

$$\mathcal{B}' = \mathcal{B}'' \cup \{y_{k-1}x_kd, cw_{l-1}z_l\}.$$

The proof that $y_{k-1}x_kd$ and $cw_{l-1}z_l$ are edge-disjoint from all of $\mathcal{B}' \cap \mathcal{B}$ is as before. Also, $w_{l-1}z_l$, $y_{k-1}x_k \notin E(P(x)) \cap E(P(z))) \setminus M_v$ and c, $d \neq v$ imply that $y_{k-1}x_kd$ and $cw_{l-1}z_l$ are edge-disjoint from $\mathcal{B}' \setminus (\mathcal{B} \cup \{y_{k-1}x_kd, cw_{l-1}z_l\})$. Finally, $y_{k-1}x_kd$ and $cw_{l-1}z_l$ share no edges since $y_{k-1}x_k$ and $w_{l-1}z_l$ are distinct independent edges in H_v. Therefore $\mathcal{B}'$ is indeed an independent set of triangles satisfying (2.1).

This completes the proof of assertion (i).

Proof of (ii). We start by showing that C is a transversal for G. Let $T = vxy \in \mathcal{T}(G)$ where $v \in V_\sigma$, $y \in V_{\sigma+1}$, and $x \in V_{\sigma+2}$, and suppose $E(T) \cap E_\sigma(\mathcal{B}) = \emptyset$, that is $xy \notin M_v$. If y were not incident with an edge of M_v, then xv would be essential for $\mathcal{B}$, and hence $x \in \bar{V}(H_v)$ and so $E(T) \cap C_2 \neq \emptyset$. Similarly we are done if x were not incident with an edge of M_v, so we may assume that zx and wy are in M_v for some $w \in V_{\sigma+2}$ and $z \in V_{\sigma+1}$. Now if wy were essential for $\mathcal{B}$, then the alternating path $P(x) = zxyw$ would show that x is v-essential, and hence $vx \in E(T) \cap C_2$. Clearly we may also assume that y is not v-essential. Finally, to show that $vy \in C_3$, we need to show that w is not v-essential, so suppose on the contrary that it is. Then there exists an alternating path $P(w) = ywy_1x_2y_2 \dots x_k$ satisfying the required conditions. Then we claim that x is v-essential with path $P(x) = zxP(w)$. The only condition we need to check is that $P(x)$ is really a path, *i.e.*, we do not have $x = x_i$ for some $i \in \{2, \dots, k\}$ or $z = y_j$ for some $j \in \{1, \dots, k-1\}$. If the second of these holds then also $x = x_{j+1}$ since $zx \in M_v$ is the only edge in M_v

incident with z. But as noted before, if w is of type 1 then x_i is also v-essential for $i = 2, \dots, k$, which would imply that x is v-essential. Similarly if w is of type 2 and $x = x_j$ for some $j \in \{2, \dots, k-1\}$ then we are finished. Finally if w is of type 2 and $x = x_k$ then $P(y) = wyx_ky_{k-1}$ shows that y is v-essential. Therefore if x and y are not v-essential then $vy \in E(T) \cap C_3$. Hence C is a transversal of $\mathcal{T}(G)$.

Now $|C| = |\mathcal{B}| + |C_2| + |C_3|$, and by (i) we have that $|C_2| = \sum_{v \in V_\sigma} |\bar{V}(H_v)|$ and $|C_3| = \sum_{v \in V_\sigma} \big(|M_v| - |\bar{V}(H_v)| - |M_v \cap \bar{E}_\sigma(\mathcal{B})|\big)$. Hence

$$|C| = |\mathcal{B}| + \sum_{v \in V_\sigma} \big(|M_v| - |M_v \cap \bar{E}_\sigma(\mathcal{B})|\big) = 2|\mathcal{B}| - \nu_\sigma(\mathcal{B}),$$

proving (ii).

This completes the proof of Lemma 2. □

Proof of Lemma 3. We start by proving that $x_j \neq w_r$ for all j and r. First note that $x \neq w_r$ for any r since $P(z)$ is a path. Now suppose that $x_j = w_r$ for some $j \in \{2, \dots, k-1\}$ and $r \in \{1, \dots, l-2\}$. Then since M_v is a matching and both $y_{j-1}x_j$ and $w_rz_{r+1} \in M_v$, we have also that $y_{j-1} = z_{r+1}$. But then $y_{j-1}x_j \in M_v$, and $P(x_j) = y_{j-1}x_jy_j \dots x_k$ and $P(y_{j-1}) = w_rz_{r+1}w_{r+1} \dots z_l$ show that x_j and y_{j-1} are v-essential. Therefore the triple $(x_jy_{j-1}, P(x_j), P(y_{j-1}))$ is a better choice than $(xz, P(x), P(z))$, which is a contradiction.

Hence $x_j = w_r$ can happen only if $j = k$ or $r = l-1$. To complete the argument we shall now need to consider several cases according to the type of x and z. First note that the above argument is also valid, *i.e.*, $P(x_j)$ and $P(y_{j-1})$ are 'valid' alternating paths if $x_j = w_r$ and any of the following hold:

- $j = k$, $r \neq l-1$, x is of type 1 and z is of either type,
- $r = l-1$, $j \neq k$, z is of type 1 and x is of either type,
- $j = k$, $r = l-1$, and both x and z are of type 1.

We next consider the case in which $j = k$, $r \neq l-1$, and x is of type 2, *i.e.*, $y_{k-1}x_k \in M_v$ is essential for $\mathcal{B}$. Then again we must have $w_rz_{r+1} = y_{k-1}x_k \in M_v$ essential for $\mathcal{B}$, and so $P'(z) = xzw_1 \dots w_rz_{r+1}$ is a valid shorter path for z, contradicting the fact that $k+l$ is minimal. Similarly, if $r = l-1$, $j \neq k$, and z is of type 2 then $w_{l-1}z_l = y_{j-1}x_j$ and $P'(x) = zxy_1x_2 \dots y_{j-1}x_j$ is a valid shorter path for x.

The remaining cases have $j = k$ and $r = l-1$, so $y_{k-1}x_k = w_{l-1}z_l$. If x is of type 2 and z is of type 1, then the triangle $vx_ky_{k-1} \in \mathcal{B}$ has two essential edges $y_{k-1}x_k$ and vz_l. Let $y_{k-1}z_la$ and vz_lb be triangles associated with these essential edges, where $v \neq a \in V_\sigma$. Then $\mathcal{B}' = \big(\mathcal{B} \setminus \{vx_ky_{k-1}\}\big) \cup \{y_{k-1}x_ka, vz_lb\}$ is an independent set of triangles of size $|\mathcal{B}|+1$ satisfying (2.1) of our lemma. Now if x is of type 1 and z is of type 2, then the same argument applies with the edges $y_{k-1}x_k$ and vx_k that are essential for $\mathcal{B}$. Therefore the last possibility is that x and y are both of type 2. We shall show that in this case $\mathcal{B}'$ as in (2.1) exists. Let $cy_{k-1}x_k = cw_{l-1}z_l$ be a triangle associated with the essential edge $y_{k-1}x_k$, so $v \neq c \in V_\sigma$. Let

$$\mathcal{B}' = \mathcal{B}'' \cup \{cy_{k-1}x_k\},$$

where $\mathcal{B}''$ is as defined in (**). Here the triangle $cy_{k-1}x_k$ is edge-disjoint from any $T \in \mathcal{B}' \setminus (\mathcal{B}'_v \cup \{cy_{k-1}x_k\}) = \mathcal{B} \setminus \mathcal{B}_v$ since $cy_{k-1}x_k$ is associated to the essential edge $y_{k-1}x_k$ of $\mathcal{B}$, and is edge-disjoint from any $T \in \mathcal{B}'_v$ by construction. Therefore $\mathcal{B}'$ is an independent set of triangles of size $|\mathcal{B}|+1$, and also the edge condition in (2.1) follows for $\mathcal{B}'$ from (**). This contradiction shows that $x_j \neq w_r$ for any j and r. A similar argument shows that $y_j \neq z_r$ for any j and r, and so $P(x)$ and $P(z)$ are vertex-disjoint apart from x and z, proving Lemma 3. □

§2. Proof of the Main Result

We now proceed to prove our main result, Theorem 4 below. In the sequel, the edge of colour σ in a triangle T is denoted by $e_\sigma(T)$.

Theorem 4. *Let G be a tripartite graph. Then $\tau(G) \leq \alpha\nu(G)$, where $\alpha = (38 + \sqrt{1948})/42 < 1.956$.*

Proof. The proof is by induction on $\nu = \nu(G)$. The case $\nu = 1$ is immediate, and hence we assume that $\nu = \nu(G) \geq 2$ and that our theorem holds for graphs G' with $\nu(G') < \nu$. Let $\mathcal{B}_0$ be a fixed independent set of triangles in G of size ν. Our first aim is to construct a second large independent set $\mathcal{B}_1$ of triangles of G. Let G' denote the graph obtained from G by deleting all edges in $E_1(\mathcal{B}_0)$, let $\mathcal{B}'$ be an independent set of triangles of G' of maximum size, and let $\nu' = |\mathcal{B}'|$.

Case 1. $\nu' = \nu$

In this case, we claim that $\tau(G) \leq 7\nu/4 < \alpha\nu$, which completes the induction step. To prove the claim, let

$$C_0 = E_2(\mathcal{B}_0) \cup E_2(\mathcal{B}') \cup \bar{E}_3(\mathcal{B}_0) \cup \bar{E}_3(\mathcal{B}')$$

and let $T \in \mathcal{T}(G)$. Suppose that $E(T) \cap (E_2(\mathcal{B}_0) \cup E_2(\mathcal{B}')) = \emptyset$. Since $\nu' = \nu$ it is clear that T must contain an edge of $E(\mathcal{B}_0)$ and an edge of $E(\mathcal{B}')$. But $E_1(\mathcal{B}_0) \cap E_1(\mathcal{B}') = \emptyset$ by construction, so for one of $\mathcal{B} = \mathcal{B}_0$ or $\mathcal{B}'$ we have $E(T) \cap E_1(\mathcal{B}) = \emptyset$. Therefore $E(T) \cap E_3(\mathcal{B}) \neq \emptyset$, and by definition T shows that $e_3(T)$ is essential for $\mathcal{B}$. Thus $E(T) \cap \bar{E}_3(\mathcal{B}) \neq \emptyset$, and so we have shown that C_0 is a transversal.

To estimate $|C_0|$, we first consider $|E_2(\mathcal{B}') \setminus E_2(\mathcal{B}_0)|$. Note that if $T \in \mathcal{B}'$ and $e_2(T) \notin E_2(\mathcal{B}_0)$ then T shows that $e_3(T)$ is essential for $\mathcal{B}_0$. Since any two such T are edge-disjoint, we see that there are at most $\nu_3(\mathcal{B}_0) = |\bar{E}(\mathcal{B}_0)|$ such triangles. Therefore $|E_2(\mathcal{B}') \setminus E_2(\mathcal{B}_0)| \leq \nu_3(\mathcal{B}_0)$, and hence $|C_0| \leq \nu + 2\nu_3(\mathcal{B}_0) + \nu_3(\mathcal{B}')$. Now, by Lemma 2, we are finished if $\nu_3(\mathcal{B}_0) \geq \nu/4$ or $\nu_3(\mathcal{B}') \geq \nu/4$, so let us assume that $\nu_3(\mathcal{B}_0)$, $\nu_3(\mathcal{B}') < \nu/4$. But then $|C_0| \leq 7\nu/4$, which proves the claim, and hence this case is finished.

Case 2. $\nu' < \nu$

In this case, by induction there exists a transversal C' of G' of size at most $\alpha\nu'$. Notice that $C' \cup E_1(\mathcal{B}_0)$ is a transversal for G of size at most $\nu + \alpha\nu'$, and so we may assume that $\nu + \alpha\nu' > \alpha\nu$, that is, that $\nu' > (\alpha - 1)\nu/\alpha$. We now consider $\mathcal{B}'$ as an independent set of triangles in G. We apply Lemma 1 repeatedly with $\sigma = 2$ and $\sigma = 3$ to obtain a sequence $\mathcal{B}' = \mathcal{B}^1, \mathcal{B}^2, \ldots, \mathcal{B}^k$ of independent families of triangles in G, where for each $i = 1, \ldots, k-1$ we have $|\mathcal{B}^{i+1}| = |\mathcal{B}^i| + 1$ and $E_1(\mathcal{B}^i) \subset E_1(\mathcal{B}^{i+1})$, and $\mathcal{B}^k$ satisfies (1.2) of Lemma 1. Let $\nu'' = |\mathcal{B}^k|$. Then by Lemma 1 we know that $\tau(G) \leq 2\nu''$, and so we may assume that $\nu'' > \alpha\nu/2$.

Finally, to obtain $\mathcal{B}_1$, we apply Lemma 2 repeatedly with $\sigma = 2$ and $\sigma = 3$ to obtain a further sequence $\mathcal{B}^k, \mathcal{B}^{k+1}, \ldots, \mathcal{B}^f = \mathcal{B}_1$ of independent families of triangles of G, where for each $i = k, \ldots, f-1$ we have $|\mathcal{B}^{i+1}| = |\mathcal{B}^i| + 1$ and $|E_1(\mathcal{B}^i) \cap E_1(\mathcal{B}^{i+1})| \geq |\mathcal{B}^i| - 1$. Then by (2.2) of Lemma 2 we have

(3) *$\tau(G) \le 2|\mathcal{B}_1| - \nu_\sigma(\mathcal{B}_1)$ for both $\sigma = 2$ and $\sigma = 3$.*

Let $\nu_1 = |\mathcal{B}_1|$. Clearly $\nu_1 \le \nu$. By construction we have that $|E_1(\mathcal{B}_1) \cap E_1(\mathcal{B}')| \ge \nu' - (\nu_1 - \nu'')$, and so

$$|E_1(\mathcal{B}_1) \cap E_1(\mathcal{B}_0)| \le \nu_1 - |E_1(\mathcal{B}_1) \cap E_1(\mathcal{B}')| \le 2\nu_1 - \nu' - \nu''. \tag{4}$$

Our aim is to choose a set C of edges from $E(\mathcal{B}_0) \cup E(\mathcal{B}_1)$ that will be a transversal of G of suitable size. We shall select C based on the following assertion.

($\dagger$) *One of the following holds:*

(a) $|E_2(\mathcal{B}_1) \setminus E_2(\mathcal{B}_0)| \le \nu_3(\mathcal{B}_0) + \frac{1}{2}\nu_1(\mathcal{B}_0) + \nu - \frac{1}{2}(\nu' + \nu'')$, *or*
(b) $|E_3(\mathcal{B}_1) \setminus E_3(\mathcal{B}_0)| \le \nu_2(\mathcal{B}_0) + \frac{1}{2}\nu_1(\mathcal{B}_0) + \nu - \frac{1}{2}(\nu' + \nu'')$.

Consider first the set $\mathcal{S}$ of triangles $T \in \mathcal{B}_1$ for which $e_1(T) \in E_1(\mathcal{B}_1) \cap E_1(\mathcal{B}_0)$. Note that clearly $|\mathcal{S}| = |E_1(\mathcal{B}_1) \cap E_1(\mathcal{B}_0)|$. If $T \in \mathcal{S}$ and both $e_2(T) \notin E_2(\mathcal{B}_0)$ and $e_3(T) \notin E_3(\mathcal{B}_0)$, then T shows that $e_1(T)$ is essential for $\mathcal{B}_0$. Therefore the number of such triangles T is at most $\nu_1(\mathcal{B}_0)$. Hence at least $|E_1(\mathcal{B}_1) \cap E_1(\mathcal{B}_0)| - \nu_1(\mathcal{B}_0)$ of the triangles $T \in \mathcal{S}$ are such that either $e_2(T) \in E_2(\mathcal{B}_0)$ or $e_3(T) \in E_3(\mathcal{B}_0)$. Thus for one of $\sigma = 2$ or $\sigma = 3$ we have that at most $\nu_1(\mathcal{B}_0) + (|E_1(\mathcal{B}_1) \cap E_1(\mathcal{B}_0)| - \nu_1(\mathcal{B}_0))/2$ triangles in $\mathcal{S}$ are such that $e_\sigma(T) \notin E_\sigma(\mathcal{B}_0)$. Let us first suppose that here $\sigma = 2$, and we proceed to prove that (a) holds. The proof for (b) is exactly the same if $\sigma = 3$.

Now we look at $\mathcal{B}_1 \setminus \mathcal{S}$. If $T \in \mathcal{B}_1 \setminus \mathcal{S}$ then $e_1(T) \notin E_1(\mathcal{B}_0)$. If also $e_2(T) \notin E_2(\mathcal{B}_0)$ then T shows that $e_3(T)$ is essential for $\mathcal{B}_0$. Thus the number of $T \in \mathcal{B}_1 \setminus \mathcal{S}$ for which $e_2(T) \notin E_2(\mathcal{B}_0)$ is at most $\nu_3(\mathcal{B}_0)$. Hence altogether the number of $T \in \mathcal{B}_1$ for which $e_2(T) \notin E_2(\mathcal{B}_0)$ is at most $\nu_3(\mathcal{B}_0) + (|E_1(\mathcal{B}_1) \cap E_1(\mathcal{B}_0)| + \nu_1(\mathcal{B}_0))/2$, that is

$$|E_2(\mathcal{B}_1) \setminus E_2(\mathcal{B}_0)| \le \nu_3(\mathcal{B}_0) + \frac{1}{2}\nu_1(\mathcal{B}_0) + \nu - \frac{1}{2}(\nu' + \nu''),$$

where we use (4). This proves (a), and hence assertion ($\dagger$).

To finish the proof of Case 2, we construct a suitable transversal C for G. In what follows, we assume without loss of generality that (a) holds in assertion ($\dagger$). Let

$$C = \bar{E}_3(\mathcal{B}_0) \cup \bar{E}_3(\mathcal{B}_1) \cup E_2(\mathcal{B}_0) \cup E_2(\mathcal{B}_1) \cup (E_1(\mathcal{B}_0) \cap E_1(\mathcal{B}_1)),$$

and we claim that C is a transversal of $\mathcal{T}(G)$. To see this, suppose that $T \in \mathcal{T}(G)$ and $E(T) \cap \{E_2(\mathcal{B}_0) \cup E_2(\mathcal{B}_1) \cup (E_1(\mathcal{B}_1) \cap E_1(\mathcal{B}_0))\} = \emptyset$. Then for one of $\mathcal{B} = \mathcal{B}_0$ or $\mathcal{B}_1$, we have that $e_1(T) \notin E_1(\mathcal{B})$ and $e_2(T) \notin E_2(\mathcal{B})$. Hence $e_3(T)$ is essential for $\mathcal{B}$, and $E(T) \cap (\bar{E}_3(\mathcal{B}_0) \cup \bar{E}_3(\mathcal{B}_1)) \ne \emptyset$, which shows that C is indeed a transversal for G.

Now it only remains to estimate $|C|$. Using (4) and assertion ($\dagger$)(a), we have

$$\begin{aligned}|C| &\le \nu_3(\mathcal{B}_0) + \nu_3(\mathcal{B}_1) + \nu \\ &\qquad + \nu_3(\mathcal{B}_0) + \frac{1}{2}\nu_1(\mathcal{B}_0) + \nu - \frac{1}{2}(\nu' + \nu'') + 2\nu - \nu' - \nu'' \\ &= 2\nu_3(\mathcal{B}_0) + \nu_3(\mathcal{B}_1) + \frac{1}{2}\nu_1(\mathcal{B}_0) + 4\nu - \frac{3}{2}(\nu' + \nu'').\end{aligned}$$

Now, by (3) and Lemma 2, for each pair $(\sigma, i) \in \{(3,1), (3,0), (1,0)\}$ there is a transversal of $\mathcal{T}(G)$ of size at most $2\nu - \nu_\sigma(\mathcal{B}_i)$. Therefore we may assume that each

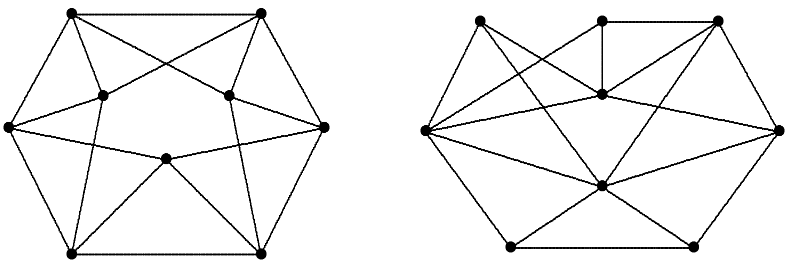

Figure 1. The graphs H_1 and H_2

of $\nu_3(\mathcal{B}_0)$, $\nu_1(\mathcal{B}_0)$, and $\nu_3(\mathcal{B}_1)$ is less than $(2-\alpha)\nu$. Further recall that $\nu'' > \alpha\nu/2$ and $\nu' > (\alpha-1)\nu/\alpha$. Hence

$$\begin{aligned} |C| &\le \frac{7}{2}(2-\alpha)\nu + 4\nu - \frac{3}{2}(\nu' + \nu'') \\ &\le \nu\left(11 - \frac{7}{2}\alpha - \frac{3}{2}\left(\frac{\alpha-1}{\alpha} + \frac{1}{2}\alpha\right)\right) = \frac{\nu}{4\alpha}(6 + 38\alpha - 17\alpha^2) \le \alpha\nu. \end{aligned}$$

This finishes the induction step in this case and the theorem is proved. □

We close with the remark that finding examples of tripartite graphs G with $\tau(G)/\nu(G)$ much larger than 1 seems difficult. The graphs H_1 and H_2 in Figure 1 are such that $\tau(G) = 5\nu(G)/4$. Therefore any graph G whose blocks are isomorphic to copies of K^2, H_1 or H_2 will be such that $\tau(G) \ge 5\nu(G)/4$.

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