MATROIDS WITH MANY COMMON BASES

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ABSTRACT. In [5], Lemos proved a conjecture of Mills [7]: for two $(k+1)$-connected matroids whose symmetric difference between their collections of bases has size at most k, there is a matroid that is obtained from one of these matroids by relaxing n_1 circuit-hyperplanes and from the other by relaxing n_2 circuit-hyperplanes, where n_1 and n_2 are non-negative integers such that $n_1 + n_2 \leq k$. In this paper, we prove a similar result, where the hypothesis of the matroids being k-connected is replaced by the weaker hypothesis of being vertically k-connected.

1. INTRODUCTION

In this article, for matroid theory we shall use the notation set by Oxley in [13], which we assume familiar to the reader. We define the *connectivity function* of a matroid M as

$$\xi_M(X, Y) = r(X) + r(Y) - r(M),$$

for a partition $\{X, Y\}$ of $E(M)$. Tutte [19] said that a matroid M is *n-connected*, for a positive integer n, provided

$$\xi_M(X, Y) \geq k$$

for every integer k and partition $\{X, Y\}$ of $E(M)$ such that $k < n$ and

$$\min\{|X|, |Y|\} \geq k.$$

Lemos [5] proved, up to small modifications, the following conjecture made by Mills [7].

Conjecture 1.1. *Let k be a positive integer. Suppose M_1 and M_2 are $(k+1)$-connected matroids on a set E where $|E| \geq 2$. If $|\mathcal{B}(M_1) \triangle \mathcal{B}(M_2)| = k$, then there is an integer $j \in [0, k]$ and a matroid N on E that is obtained from M_1 and M_2 by relaxing j and $k - j$ circuit-hyperplanes, respectively.*

Observe that, for $r \geq 1$, $U_{r-1,2r-1}$ and $U_{r,2r-1}$ are m-connected matroids for every positive integer m. As these matroids have different rank, it follows that

$$|\mathcal{B}(U_{r-1,2r-1}) \triangle \mathcal{B}(U_{r,2r-1})| = |\mathcal{B}(U_{r-1,2r-1}) \cup \mathcal{B}(U_{r,2r-1})| = 2\binom{2r-1}{r} = f(r).$$

When $k = f(r)$, this pair of matroids is a counter-example to Conjecture 1.1. Lemos [5] proved that these are the only exceptions and that the hypothesis on the size of E can be removed from this conjecture.

In [7], Mills proved this conjecture for $k = 2$ and pointed out that it follows from Truemper's results when $k = 1$ [17]. In general, this conjecture follows from

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a theorem which guarantees the same conclusion with weaker hypotheses. To state this result, we need some definitions.

For a matroid M, the *girth* of M is defined as

$$\mathrm{g}(M) = \min\{|C| : C \in \mathcal{C}(M)\},$$

where this minimum is taken to be 0, when $\mathcal{C}(M) = \emptyset$. We say that (M, N) is a k-pair, for a non-negative integer k, when N is a matroid having the same ground set as M and $|\mathcal{B}(M) \triangle \mathcal{B}(N)| \leq k$. Observe that this definition is invariant under duality; that is, (M, N) is a k-pair if and only if (M^*, N^*) is a k-pair. Lemos [5] proved:

Theorem 1.1. *Let k be a positive integer. If (M_1, M_2) is a k-pair and*

$$\min\{\mathrm{g}(M_1), \mathrm{g}(M_1^*)\} \geq k + 1,$$

then there is a matroid N which is obtained from M_1 and M_2 by relaxing n_1 and n_2 circuit-hyperplanes, respectively, where n_1 and n_2 are non-negative integers such that

$$n_1 + n_2 = |\mathcal{B}(M_1) \triangle \mathcal{B}(M_2)| \leq k.$$

In this theorem, if we replace the hypothesis of (M_1, M_2) to be a k-pair by to be a l-pair, for some $l > k$, then the conclusion does not hold, as we shall see in the next example. Let E and A be an n-element set and a r-element set, respectively, such that $A \subseteq E$, where $2r > n$. Choose $a \in A$. Suppose that M_1 and M_2 are matroids over E having every r-element set as a basis with the exception of these that contains A or $A \setminus \{a\}$, respectively. Observe that

$$g(M_1) = r, g(M_2) = r - 1 \text{ and } g(M_1^*) = g(M_2^*) = n - r.$$

So, M_1 and M_2 are $(n-r)$-connected, since $n - r < r$, by hypothesis. We have that

$$\mathcal{B}(M_1) \triangle \mathcal{B}(M_2) = \{(A \setminus \{a\}) \cup \{b\} : b \in E \setminus A\}.$$

Hence (M_1, M_2) is a $(n-k)$-pair.

An n-connected matroid M such that $|E(M)| \geq 2n - 2$ must satisfy

$$\min\{g(M), g(M^*)\} \geq n. \tag{1.1}$$

As a consequence of the previous result, Lemos [5] obtained the following theorem that proves Mills's Conjecture.

Theorem 1.2. *For some positive integer k, suppose that M_1 and M_2 are $(k+1)$-connected matroids having the same ground set E. If $|\mathcal{B}(M_1) \triangle \mathcal{B}(M_2)| = k$, then either*

(a) *there is an integer $j \in [0, k]$ and a matroid N on E that is obtained from M_1 and M_2 by relaxing j and $k - j$ circuits-hyperplanes, respectively; or*
(b) *there is a positive integer r such that*

$$2\binom{2r-1}{r} = k$$

and $\{M_1, M_2\} = \{U_{r-1,2r-1}, U_{r,2r-1}\}$.

Condition (1.1) is very strong. This happens because the definition of n-connected matroid proposed by Tutte has the attractive property of being invariant under duality but it does not generalize the notion of an n-connected graph. To generalize it, we need a new definition (see [9, 10]). We say that a matroid M is *vertically n-connected*, for a positive integer n, provided

$$\xi_M(X,Y) \geq k$$

for every integer k and partition $\{X,Y\}$ of $E(M)$ such that $k<n$ and

$$\min\{r(X), r(Y)\} \geq k.$$

Observe that every vertically n-connected matroid is n-connected, but the reverse does not occur. When M is a vertically n-connected matroid such that $r(M) \geq n$, then

$$\mathrm{g}(M^*) \geq n. \tag{1.2}$$

Note that this condition is much weaker than (1.1).

To state the main result of this paper, we need to describe an operation that generalizes the relaxation of a circuit-hyperplane. We say that a hyperplane H of a matroid M is a *tip-hyperplane* having *tip* e provided $e \in H, r(M) \leq |H|$ and $M|H$ is obtained from $M|(H \setminus \{e\})$ by adding e freely. Hence

$$\mathcal{C}_e(M|H) = \{B \cup \{e\} : B \in \mathcal{B}(M|(H \setminus \{e\}))\}.$$

In particular, every circuit of $M|H$ that contains e has size equal to $r(M)$. In section three, we shall prove that $\mathcal{B}(M) \cup \mathcal{C}_e(M|H)$ is the set of bases of a matroid which we say that is *obtained from M by relaxing the tip-hyperplane H*.

Theorem 1.3. *For some positive integer k, suppose that M_1 and M_2 are vertically $(k+1)$-connected matroids having the same ground set E. If $|\mathcal{B}(M_1) \triangle \mathcal{B}(M_2)| \leq k$, then:*

(i) *there is a matroid N on E that is obtained from each of M_1 and M_2 by relaxing a sequence of tip-hyperplanes. Moreover,*

$$\mathcal{B}(M_1) \triangle \mathcal{B}(M_2) = [\mathcal{B}(N) \setminus \mathcal{B}(M_1)] \cup [\mathcal{B}(N) \setminus \mathcal{B}(M_2)]; \ \textit{or}$$

(ii) *for each $i \in \{1,2\}$, $r(M_i) \leq k$ and M_i does not have disjoint cocircuits.*

Suppose that M_1 and M_2 are matroids over the same ground set. When M_i does not have disjoint cocircuits, for each $i \in \{1,2\}$, then M_i is a vertically n-connected matroid for every n. So, the pairs of matroids described in (ii) appear because we do not ask in the definition of a vertically n-connected matroid that its rank is at least n — that is, we permit small matroids to be vertically n-connected.

Observe that Theorem 1.2 is a consequence of this theorem.

2. Preliminary results

By (1.2), we need results about k-pairs having a matroid on it with large cogirth. We obtain it by combining Lemmas 1 and 2 of [5] in the following result:

Lemma 2.1. *Let k be a positive integer. If (M_1, M_2) is a k-pair and $\mathrm{g}(M_1^*) \geq k+1$, then*

(i) *$r(M_1) = r(M_2)$; and*

(ii) *$\mathcal{I}(M_1) \cap \mathcal{C}(M_2) \subseteq \mathcal{B}(M_1)$.*

Lemma 2.2. *Let k be a positive integer. If (M_1, M_2) is a k-pair and*

$$\min\{\mathrm{g}(M_1^*), \mathrm{g}(M_2^*)\} \geq k+1,$$

then

$$\mathcal{C} = \mathcal{B}(M_1) \bigtriangleup \mathcal{B}(M_2),$$

where $\mathcal{C}$ is the set of minimal elements belonging to $\mathcal{C}(M_1) \bigtriangleup \mathcal{C}(M_2)$.

Proof. When $C \in \mathcal{C}$, there is an $i \in \{1, 2\}$ such that $C \in \mathcal{C}(M_i) \cap \mathcal{I}(M_{3-i})$. Thus, by Lemma 2.1(ii), we have that $C \in \mathcal{B}(M_{3-i})$. Hence

$$\mathcal{C} \subseteq \mathcal{B}(M_1) \bigtriangleup \mathcal{B}(M_2). \tag{2.1}$$

Now, let B be an element of $\mathcal{B}(M_1) \bigtriangleup \mathcal{B}(M_2)$. So, there is an $i \in \{1, 2\}$ such that $B \in \mathcal{B}(M_i)$ and, by Lemma 2.1(i), B contains a circuit C of M_{3-i}. Thus, $C \in \mathcal{C}$. Hence, by (2.1) and Lemma 2.1(i), $C = B$. So,

$$\mathcal{C} = \mathcal{B}(M_1) \bigtriangleup \mathcal{B}(M_2)$$

and the result follows. □

For a matroid M, a subset L of $E(M)$ is said to be a *Tutte-line*, when $M|L$ has corank equal to two and no coloops. In [18], Tutte proved that L has a partition, which we call *the canonical partition of L in M*, $\{P_1, P_2, \dots, P_n\}$, for some $n \geq 2$, such that $\mathcal{C}(M|L) = \{L \setminus P_1, L \setminus P_2, \dots, L \setminus P_n\}$. We denote by $\mathcal{TL}(M)$ the set of Tutte-lines of the matroid M. Following Tutte [18], we say that a set of circuits $\mathcal{L}$ of a matroid M is a *linear subclass* of M, when $\mathcal{C}(M|L) \subseteq \mathcal{L}$ for every $L \in \mathcal{TL}(M)$ such that $|\mathcal{C}(M|L) \cap \mathcal{L}| \geq 2$.

By Lemma 2.2, the hypotheses of the next lemma is satisfied provided (M_1, M_2) is a k-pair such that $\min\{\mathrm{g}(M_1^*), \mathrm{g}(M_2^*)\} \geq k+1$. This also occurs, when M_2 is obtained from M_1 by relaxing some circuit-hyperplanes, or more generally, when M_2 is obtained from M_1 by transforming a special set of circuits each having $r(M_1)$ elements into bases.

Lemma 2.3. *Let M_1 and M_2 be matroids having the same rank and ground set such that*

$$\mathcal{C} = \mathcal{B}(M_1) \bigtriangleup \mathcal{B}(M_2),$$

where $\mathcal{C}$ is the set of minimal elements belonging to $\mathcal{C}(M_1) \triangle \mathcal{C}(M_2)$. If $C \in \mathcal{C} \cap \mathcal{C}(M_1)$ and $Z = \mathrm{cl}_{M_1}(C)$, then

(i) *If L is a Tutte-line of M_1 such that $L \subseteq Z$ and $\mathcal{C} \cap \mathcal{C}(M_1|L) \neq \emptyset$, then L contains just one circuit C_L of M_2 and*

$$\mathcal{C}(M_1|L) \setminus \{C_L\} = \{L \setminus \{a\} : a \in C_L\} \subseteq \mathcal{C}.$$

Moreover, $C_L \in \mathcal{C}(M_1)$ provided $C_L \neq L$.

(ii) *$\mathcal{C}(M_1|Z) \setminus \mathcal{C}$ is a linear subclass of $M_1|Z$.*

(iii) *$\{D \in \mathcal{C}(M_2) \setminus \mathcal{C}(M_1) : D \cap Z \notin \mathcal{I}(M_1)\} = \mathcal{C}_1 \cup \mathcal{C}_2$, where*
$\mathcal{C}_1 = \{L \in \mathcal{TL}(M_1|Z) : \mathcal{C}(M_1|L) \subseteq \mathcal{C}\}$
$\mathcal{C}_2 = \{D \cup \{e\} : D \in \mathcal{C} \cap \mathcal{C}(M_1|Z), e \in E(M_1) \setminus Z$ and $\mathcal{C} \cap \mathcal{C}(M_2|(D \cup \{e\})) = \emptyset\}$.

Proof. (i) Let C' a circuit belonging to $\mathcal{C}(M_1|L) \cap \mathcal{C}$. Note that $\mathrm{cl}_{M_1}(C') = Z$. As C is a basis of M_2, it follows that $r_{M_2}(L) = r_{M_1}(L) + 1 = |L| - 1$. Thus, L contains exactly one circuit C_L of M_2. We have two possibilities for C_L.

First, suppose that C_L is also a circuit of M_1. If $C' \in \mathcal{C}(M_1|L) \setminus \{C_L\}$, then C' is independent in M_2 because $C_L \nsubseteq C' \subset L$. So, $C' \in \mathcal{C}$. Hence $|C'| = r(M_1) = |L| - 1$ and $C' = L \setminus \{a\}$, for some $a \in C_L$. Thus, $\mathcal{C}(M_1|L) = \{C_L\} \cup \{L \setminus \{a\} : a \in C_L\}$.

Next, suppose that C_L is not a circuit of M_1. If $C' \in \mathcal{C}(M_1|L)$ and $|C'| < r(M_1)$, then, by hypothesis, C' is also a circuit of M_2 and so $C' = C_L$. We have a contradiction and hence $|C'| \geq r(M_1) = |L| - 1$, for every circuit C' of $M_1|L$. Thus, $|C'| = |L| - 1$, for every circuit C' of $M_1|L$. In particular, $\mathcal{C}(M_1|L) = \{L \setminus \{a\} : a \in L\}$. As C_L is not a circuit of M_1, it follows that $|C_L| \geq r(M_1)$, by hypothesis. Hence $C_L = L$.

(ii) If L is a Tutte-line of $M_1|Z$ containing at least one circuit that belongs to $\mathcal{C}$, then, by (i), it contains at most one circuit that does not belong to $\mathcal{C}$ which is C_L when $C_L \neq L$. By definition, $\mathcal{L} = \mathcal{C}(M_1|Z) \setminus \mathcal{C}$ is a linear subclass of $M_1|Z$.

(iii) If $C' \in \{D \in \mathcal{C}(M_2) \setminus \mathcal{C}(M_1) : D \cap Z \notin \mathcal{I}(M_1)\}$, then C' contains properly a circuit C'' of $M_1|Z$. So, $C'' \in \mathcal{C}$ and $|C'| = |C''| + 1$. When $C' \subseteq Z$, then C' is a Tutte-line of $M_1|Z$ and, by (i), $\mathcal{C}(M_1|C') \subseteq \mathcal{C}$. So, $C' \in \mathcal{C}_1$. Thus, we may suppose that $C' \nsubseteq Z$. As $C'' \subseteq Z \cap C'$ and $|C'| = |C''| + 1$, it follows that $C' \setminus Z = \{e\}$. Hence $C' = C'' \cup \{e\}$ and $C' \in \mathcal{C}_2$.

Next, we prove that $C'' \cup \{e\}$ is a circuit of M_2, for every $e \notin Z$ and $C'' \in \mathcal{C} \cap \mathcal{C}(M_1|Z)$ such that $\mathcal{C}(M_2|(C'' \cup \{e\})) \cap \mathcal{C} = \emptyset$. This happens because $C'' \cup \{e\}$ is dependent in M_2, C'' is a basis of M_2, every circuit of $M_2|(C'' \cup \{e\})$ having at most $r(M_1) = r(M_2)$ elements is also a circuit of M_1 and Z does not span e in M_1. So, every set belonging to $\mathcal{C}_2$ is a circuit of M_2. With similar arguments, we conclude that every set belonging to $\mathcal{C}_1$ is a circuit of M_1 and the result follows. $\square$

3. A generalization of the relaxation

We say that a circuit C of a matroid M is *large*, when $r(M) = |C|$. Let $\mathcal{LC}(M)$ and $\mathcal{CH}(M)$ be the sets of large circuits of M and circuit-hyperplanes of M, respectively. Thus, $\mathcal{CH}(M) \subseteq \mathcal{LC}(M)$. Kahn proved that, for each $\mathcal{X} \subseteq \mathcal{CH}(M)$, there is a matroid $M_{\mathcal{X}}$ such that

$$\mathcal{B}(M_{\mathcal{X}}) = \mathcal{B}(M) \cup \mathcal{X}.$$

We say that $M_{\mathcal{X}}$ is obtained from M by *relaxing the circuit-hyperplanes belonging to* $\mathcal{X}$. Moreover,

$$\mathcal{CH}(M_{\mathcal{X}}) = \mathcal{CH}(M) \setminus \mathcal{X}.$$

By this identity, one can arrive at the matroid $M_{\mathcal{X}}$ by relaxing the circuit-hyperplanes belonging to $\mathcal{X}$ one at each time — this construction is independent of the chosen order for relaxing the circuit-hyperplanes. To see the relevance of this construction we refer, for example, to [3, 4, 6, 7, 8, 11, 12, 14, 16, 19]. In this section, we shall discuss an extension of this construction, where we take $\mathcal{X}$ to be a set of large circuits of M which must satisfies a special condition. In the previous subsection, we obtained the necessary conditions that $\mathcal{X}$ must satisfy and, in this section, we shall prove that these conditions are also sufficient.

We say that Z is a *nest* of a matroid M, when $Z = \mathrm{cl}_M(C)$, for some $C \in \mathcal{LC}(M)$. Observe that $Z = \mathrm{cl}_M(C')$, for every $C' \in \mathcal{LC}(M)$ such that $C' \subseteq Z$. When a circuit-hyperplane C'' of M is contained in a nest Z of M, then $Z = C''$.

Suppose that C is a large circuit of a matroid M. Let Z be the nest of M that contains C. In this subsection, we shall consider a liner subclass $\mathcal{L}$ of $M|Z$ satisfying

$$\mathcal{L} \neq \mathcal{C}(M|Z) \text{ and } \mathcal{C}(M|Z) \setminus \mathcal{HAM}(M|Z) \subseteq \mathcal{L},$$

where $\mathcal{HAM}(H)$ denotes the set of Hamiltonian circuits of a matroid H. (The existence of Hamiltonian circuits in a matroid is very important: nice applications of them can be found in [1, 2].) A linear subclass of $M|Z$ satisfying these conditions will be called *admissible*. Now, we shall construct a family of sets, similar to the family described in Lemma 2.3(iii), and prove that it is a family of circuits of a matroid. We set

$$\begin{aligned}\mathcal{S}_{\mathcal{L}} &= \{C' \cup \{e\} : C' \in \mathcal{C}(M|Z) \setminus \mathcal{L} \text{ and } e \in E(M) \setminus Z\}\\ \mathcal{T}_{\mathcal{L}} &= \{L : L \in \mathcal{TL}(M|Z) \text{ and } \mathcal{C}(M|L) \cap \mathcal{L} = \emptyset\}\end{aligned}$$

We shall prove that

$$\mathcal{C}_{\mathcal{L}} = [\mathcal{C}(M) \setminus (\mathcal{C}(M|Z) \setminus \mathcal{L})] \cup \mathcal{S}_{\mathcal{L}} \cup \mathcal{T}_{\mathcal{L}}$$

is the set of circuits of a matroid, which we shall denote by $M_{Z,\mathcal{L}}$. We say that this matroid is obtained from M by *relaxing the nest Z along the admissible linear subclass $\mathcal{L}$ of $M|Z$*. Thus, $\mathcal{B}(M_{Z,\mathcal{L}}) = \mathcal{B}(M) \cup (\mathcal{C}(M|Z) \setminus \mathcal{L})$.

Suppose that C_1 and C_2 are different sets belonging to $\mathcal{C}_{\mathcal{L}}$. We need to show only that

$$(C_1 \cup C_2) \setminus \{e\}$$

contains an element of this family, for every $e \in C_1 \cap C_2$. First, we shall prove a lemma.

Lemma 3.1. *If $X \subseteq E(M)$ and $|X| = r(M) + 1$, then X contains a member of $\mathcal{C}_{\mathcal{L}}$.*

Proof. Observe that X contains a circuit C' of M. If C' also belongs to $\mathcal{C}_{\mathcal{L}}$, then the result follows. So, we may suppose that C' does not belong to this family. Thus, $C' \in \mathcal{C}(M|Z) \setminus \mathcal{L}$. In particular, $|X \setminus Z| \le 1$. When $X \setminus Z = \{e\}$, then $X = C' \cup \{e\}$ and X belongs to $\mathcal{S}_{\mathcal{L}}$. So, we may suppose that X is contained in Z. In this case, X is a Tutte-line of $M|Z$. Moreover, we may suppose that every circuit of M contained in X belongs to $\mathcal{C}(M|Z) \setminus \mathcal{L}$, otherwise the result follows. Hence X belongs to $\mathcal{T}_{\mathcal{L}}$ and the result follows. □

By Lemma 3.1, we may suppose that $|C_1 \cup C_2| \le r(M)+1$, otherwise $(C_1 \cup C_2) \setminus \{e\}$ has at least $r(M) + 1$ elements and contains an element of $\mathcal{C}_{\mathcal{L}}$. As C_1 and C_2 are different, it follows that $|C_i| \le r(M)$, for every $i \in \{1, 2\}$. Hence

$$\{C_1, C_2\} \subseteq \mathcal{L} \cup (\mathcal{C}(M) \setminus \mathcal{C}(M|Z)).$$

So, there is circuit C_3 of M such that

$$C_3 \subseteq (C_1 \cup C_2) \setminus \{e\}.$$

If $|C_3| < r(M)$, then C_3 belongs to $\mathcal{C}_{\mathcal{L}}$ because $\mathcal{C}(M|Z) \setminus \mathcal{HAM}(M|Z) \subseteq \mathcal{L}$. So, we may suppose that $|C_3| = r(M)$ and that $C_3 \in \mathcal{C}(M|Z) \setminus \mathcal{L}$. Hence $C_1 \cup C_2 = C_3 \cup \{e\}$ is a Tutte-line. As $C_3 \subseteq Z$, it follows that $C_1 \cup C_2 \subseteq Z$, since Z is a closed set in M. In particular, $\{C_1, C_2\} \subseteq \mathcal{L}$. By the definition of a linear subclass, we have that $C_3 \in \mathcal{L}$ because both C_1 and C_2 belongs to $\mathcal{L}$. We have a contradiction. Thus, $\mathcal{C}_{\mathcal{L}}$ is a family of circuits of a matroid that is different of M because $\mathcal{L} \neq \mathcal{C}(M|Z)$.

When $\max\{|C'| : C' \in \mathcal{L}\} < r(M)$, then

$$\mathcal{L} = \mathcal{C}(M|Z) \setminus \mathcal{HAM}(M|Z) = \mathcal{C}(M_{Z,\mathcal{L}}|Z) \setminus \mathcal{HAM}(M_{Z,\mathcal{L}}|Z). \tag{3.1}$$

That is, $M_{Z,\mathcal{L}}|Z$ and $M|Z$ have the same non-spanning circuits. Matroids having the same non-spanning circuits have been considered by Oxley an Wittle in [15]. We are not using theirs results here, since, there are cases where the condition stated at the beginning of this paragraph does not hold and (3.1) is not true.

The next lemma has a straitforward proof. So, it will be omitted. In it, we show that loops are irrelevant to our construction.

Lemma 3.2. *Suppose that $r(M) \geq 2$. If l is a loop of $M|Z$, then l is a loop of $M_{Z,\mathcal{L}}$. Moreover,*

(i) *$Z \setminus \{l\}$ is a nest of $M\backslash l$.*
(ii) *$\mathcal{L} \setminus \{\{l\}\}$ is an admissible linear subclass of $(M\backslash l)|(Z \setminus \{l\})$.*
(iii) *$M_{Z,\mathcal{L}}\backslash l = (M\backslash l)_{Z\setminus\{l\},\mathcal{L}\setminus\{\{l\}\}}$.*

Now, we shall prove the last result of this section.

Lemma 3.3. *If Z' is a nest of M different of Z, then Z' is a nest of $M_{Z,\mathcal{L}}$. Moreover, $M|Z' = M_{Z,\mathcal{L}}|Z'$.*

Proof. First, we shall prove that

$$\mathcal{LC}(M_{Z,\mathcal{L}}) = \mathcal{LC}(M) \setminus (\mathcal{C}(M|Z) \setminus \mathcal{L}). \tag{3.2}$$

As $r(Z \cap Z') < r(M) - 1$, it follows that every large circuit of M contained in Z' has at most $r(M) - 2$ elements belonging to Z. That is, this circuit is a large circuit of $M_{Z,\mathcal{L}}$. Thus, (3.2) holds because every large circuit of M that do not belong to $\mathcal{C}(M|Z)$ is contained in a nest of M different of Z.

Observe that every element of $\mathcal{S}_{\mathcal{L}}$ intersects Z' in at most $r(M) - 1$ elements and every element of $\mathcal{T}_{\mathcal{L}}$ instersects Z' in at most $r(M) - 2$ elements. Hence $|C \setminus Z'| \geq 2$, for every $C \in \mathcal{S}_{\mathcal{L}} \cup \mathcal{T}_{\mathcal{L}}$. So, $\mathcal{C}(M|Z') = \mathcal{C}(M_{Z,\mathcal{L}}|Z')$ and Z' is closed in $M_{Z,\mathcal{L}}$. The result follows. ☐

This lemma is very important because we can fix a set of nests of M and for each nest in this family we can choose an admissible liner subclass of M restricted to this nest, when it exists. When we relax one of these nests along its admissible linear subclass, the other nests are nests of the resulting matroid and the admissible linear subclasses associated with them retains this property in the new matroid. So, we can continue with the process (and the order is irrelevant). Thus, this construction behaves similarly as the relaxing of a set of circuits-hyperplanes. Moreover, it agrees with it when every nest is a circuit-hyperplane.

4. A result similar to Theorem 1.1

In this section, we shall prove a result that plays a similar role in the proof of Theorem 1.3 as did Theorem 1.1 in the proof of Theorem 1.2.

Theorem 4.1. *Let k be a positive integer. If (M_1, M_2) is a k-pair and*

$$\min\{\mathrm{g}(M_1^*), \mathrm{g}(M_2^*)\} \geq k + 1,$$

then there is a matroid N which is obtained from M_i, for both $i \in \{1, 2\}$, by relaxing a sequence of nests (each along an admissible linear subclass of M_i restricted to it) such that

$$\mathcal{B}(M_1) \triangle \mathcal{B}(M_2) = (\mathcal{B}(N) \setminus \mathcal{B}(M_1)) \cup (\mathcal{B}(N) \setminus \mathcal{B}(M_2)).$$

Proof. We prove this result by induction on $|\mathcal{B}(M_1) \triangle \mathcal{B}(M_2)|$. When $|\mathcal{B}(M_1) \triangle \mathcal{B}(M_2)| = 0$, we have that $M_1 = M_2$. In this case, we take $N = M_1 = M_2$ and the result follows. So, we may suppose that $|\mathcal{B}(M_1) \triangle \mathcal{B}(M_2)| > 0$. By Lemma 2.2, $\mathcal{C} = \mathcal{B}(M_1) \triangle \mathcal{B}(M_2)$, where $\mathcal{C}$ is the set of minimal elements belonging to $\mathcal{C}(M_1) \triangle \mathcal{C}(M_2)$. In particular, $\mathcal{C} \neq 0$. Choose $C \in \mathcal{C}$. Without loss of generality, we may suppose that $C \in \mathcal{C}(M_1)$. Set $Z = \mathrm{cl}_{M_1}(C)$. By Lemma 2.3, $\mathcal{L} = \mathcal{C}(M_1|Z) \setminus \mathcal{C}$ is a linear subclass of $M_1|Z$. Observe that this linear subclass is admissible in $M_1|Z$. Let N_1 be the matroid obtained from M_1 by relaxing the nest Z along the linear subclass $\mathcal{L}$ of $M_1|Z$. Thus,

$$\mathcal{B}(N_1) = \mathcal{B}(M_1) \cup (\mathcal{C}(M_1|Z) \setminus \mathcal{L}) = \mathcal{B}(M_1) \cup (\mathcal{C} \cap \mathcal{C}(M_1|Z)).$$

As $\mathcal{C} = \mathcal{B}(M_1) \triangle \mathcal{B}(M_2)$, it follows that $\mathcal{C} \cap \mathcal{C}(M_1|Z) \subseteq \mathcal{B}(M_2) \setminus \mathcal{B}(M_1)$ and so

$$\mathcal{B}(N_1) \triangle \mathcal{B}(M_2) = [\mathcal{B}(M_1) \triangle \mathcal{B}(M_2)] \setminus [\mathcal{C} \cap \mathcal{C}(M_1|Z)].$$

The result follows by induction because $|\mathcal{B}(N_1) \triangle \mathcal{B}(M_2)| < |\mathcal{B}(M_1) \triangle \mathcal{B}(M_2)|$ and $\mathcal{B}(N_1) \setminus \mathcal{B}(M_1) = \mathcal{C} \cap \mathcal{C}(M_1|Z)$. □

Next, we will prove a result similar to Theorem 1.2 for vertically $(k+1)$-connected matroids, namely Theorem 1.3. This result will be a consequence of Theorem 4.1 and a lemma that will be proved in the next section: it describes the structure of the admissible linear subclasses of M_i restricted to a nest, when (M_1, M_2) is a k-pair that satisfies the hypothesis of Theorem 4.1. The linear subclasses are quite simple: it is the set of circuits contained in the nest that do not contain an element. This is a very strong structural result, since linear subclasses are closed related to one-element lifts of matroids, as we will explain in the next paragraph.

When M is a matroid and $\mathcal{L}$ is a linear subclass of M, we can obtain an one-element lift of M using this linear subclass. Choose $e \notin E(M)$. Consider the following family of sets:

$$\mathcal{L} \cup \{C \cup \{e\} : C \in \mathcal{C}(M) \setminus \mathcal{L}\} \cup \{L \in \mathcal{TL}(M) : \mathcal{C}(M|L) \cap \mathcal{L} = \emptyset\}.$$

This family is a family of circuits of a matroid N such that $N/e = M$. Moreover, any one-element lift of M is obtained in this way. Thus, when this linear subclass is the set of circuits of M that do not contain an element, say f, then the lift of M that we obtain using this linear subclass is the one obtained from M by inserting e in series with f (when f is not a coloop of M).

5. Proof of the main result

Lemma 5.1. *Let C be a large circuit of a matroid M. If $Z = \mathrm{cl}_M(C)$ and $\mathcal{L}$ is an admissible linear subclass of $M|Z$ such that*

$$|\mathcal{C}(M|Z) \setminus \mathcal{L}| \leq |C|,$$

then there is a series class S of $M|Z$ such that $\mathcal{C}(M|(Z \setminus S)) = \mathcal{L}$.

Proof. When $|Z \setminus C| = 0$, we take $S = Z$. When $|Z \setminus C| = 1$, then Z is a Tutte-line of M. If Z does not contain a circuit of $\mathcal{L}$, then every circuit of M contained in Z is large. Thus, $\mathcal{C}(M|Z) = \{Z \setminus \{a\} : a \in Z\}$. So,

$$|C| + 1 = |Z| = |\mathcal{C}(M|Z)| = |\mathcal{C}(M|Z) \setminus \mathcal{L}| \leq |C|,$$

where the last inequality follows by hypothesis. We arrive at a contradiction. Hence Z contains a circuit D of $\mathcal{L}$. Observe that $S = Z \setminus D$ is a series class of $M|Z$ with the desired property. So, we may suppose that $|Z \setminus C| \geq 2$.

For each $z \in Z \setminus C$, $C \cup \{z\}$ is a Tutte-line of M. By the first paragraph, let C_z be the circuit of $C \cup \{z\}$ that belongs to $\mathcal{L}$. By Lemma 2.3(i), we have that $(C \cup \{z\}) \setminus \{w\}$ is a circuit of $M|Z$ that does not belong to $\mathcal{L}$, for every $w \in C_z$. Set

$$\mathcal{C}_z = \{(C \cup \{z\}) \setminus \{w\} : w \in C_z \setminus \{z\}\}.$$

Observe that every circuit belonging to $\mathcal{C}_z$ contains exactly one element not belonging to C, namely z. Hence $\mathcal{C}_z \cap \mathcal{C}_w = \emptyset$, when $z \neq w$. Thus,

$$|\mathcal{C}(M|Z) \setminus \mathcal{L}| \geq |\{C\}| + \sum_{z \in Z \setminus C} |\mathcal{C}_z| = 1 + \sum_{z \in Z \setminus C} |C \cap C_z|.$$

By hypothesis, we have that

$$|C| \geq |\mathcal{C}(M|Z) \setminus \mathcal{L}| \geq 1 + \sum_{z \in Z \setminus C} |C \cap C_z| \geq 1 + \left| \bigcup_{z \in Z \setminus C} C \cap C_z \right|.$$

In particular, we have that

$$\left| \bigcup_{z \in Z \setminus C} C \cap C_z \right| \leq |C| - 1.$$

So,

$$W = \bigcup_{z \in Z \setminus C} C \cap C_z$$

is a proper subset of C. As $C \setminus W$ is contained in a series class of $M|C$ and W spans $Z \setminus C$ in $M|Z$, it follows that $C \setminus W$ is contained in a series class S of $M|Z$. Note that $S = C \setminus W$ because every element of W belongs C_z, for some $z \in Z \setminus C$, and so it could not belong to the series class S.

Now, we shall prove that $\mathcal{L} \subseteq \mathcal{C}(M|(Z \setminus S))$. Suppose that $D \in \mathcal{L} \setminus \mathcal{C}(M|(Z \setminus S))$. So, $S \subseteq D$. By Lemma 3.43 of [20], there is a Tutte-line L of $M|Z$ such that $D \subset L$ and $\mathcal{C}(M|L) \cap \mathcal{L} = \{D\}$. Hence, by Lemma 2.3(i),

$$\mathcal{C} = \{L \setminus \{d\} : d \in D\} \subseteq \mathcal{C}(M|Z) \setminus \mathcal{L}.$$

As $\mathcal{C}$ is disjoint of $\mathcal{C}_z$, since the circuits in the first set contains S and the circuits in the second set does not contain S, for every $z \in Z \setminus C$, it follows that

$$|\mathcal{C}(M|Z) \setminus \mathcal{L}| \geq |\{C\}| + |\mathcal{C}| + \sum_{z \in Z \setminus C} |\mathcal{C}_z| = 1 + |D| + \sum_{z \in Z \setminus C} |C \cap C_z|.$$

We arrive at a contradiction, by hypothesis, because

$$|\mathcal{C}(M|Z) \setminus \mathcal{L}| \geq 1 + |D| + |W| \geq 1 + |S| + |W| = 1 + |C|.$$

So, $\mathcal{L} \subseteq \mathcal{C}(M|(Z \setminus S))$.

The result follows provided we prove that $\mathcal{C}(M|(Z \setminus S)) \subseteq \mathcal{L}$. Suppose that D is a circuit of $M|(Z \setminus S)$ such that $D \notin \mathcal{L}$. By Lemma 3.43 of [20] applied to the linear subclass $\mathcal{C}(M|(Z \setminus S))$, there is a Tutte-line L of $M|Z$ such that $D \cup S \subseteq L$. As every circuit of $M|L$ different of D contains S and no circuit of $M|Z$ that contains S belongs to $\mathcal{L}$, it follows that no circuit of $M|L$ belongs to $\mathcal{L}$. We arrive at a contradiction by the first paragraph and the result follows. □

Proof of Theorem 1.3. We shall divide the proof of this result in two cases:

Case 1. $\max\{r(M_1), r(M_2)\} \leq k$.

First, we shall prove that M_i does not have disjoint cocircuits, for both $i \in \{1,2\}$. Let C_1^* and C_2^* be disjoint cocircuits of M_i, say $i = 1$. Consider the hyperplanes $H_1 = E \setminus C_1^*$ and $H_2 = E \setminus C_2^*$. Observe that

$$\begin{aligned} \xi_{M_1}(C_1^*, H_1) &= r_{M_1}(C_1^*) + r_{M_1}(H_1) - r(M_1) \\ &= r_{M_1}(C_1^*) + [r(M_1) - 1] - r(M_1) \\ &= r_{M_1}(C_1^*) - 1 \end{aligned}$$

We also have that $C_1^* \subseteq H_2$ and so $r_{M_1}(C_1^*) \leq r_{M_1}(H_2) = r(M_1) - 1$. Hence

$$\min\{r_{M_1}(C_1^*), r_{M_1}(H_1)\} \geq r_{M_1}(C_1^*).$$

We arrive at a contradiction because $r_{M_1}(C_1^*) \leq k$ and M_1 is $(k+1)$-connected. Thus, M_i does not have disjoint cocircuits, for both $i \in \{1,2\}$. In particular, M_1 and M_2 are vertically n-connected for every positive integer n. We have (ii).

Case 2. $\max\{r(M_1), r(M_2)\} \geq k+1$.

Without loss of generality, we may suppose that $r(M_1) \geq k+1$. Let C_1^* be a cocircuit of M_1 and $H_1 = E \setminus C_1^*$. Similarly to the previous case, we have that $\xi_{M_1}(C_1^*, H_1) = r_{M_1}(C_1^*) - 1$. Moreover,

$$\min\{r_{M_1}(C_1^*), r_{M_1}(H_1)\} = \min\{r_{M_1}(C_1^*), r(M_1) - 1\} \geq \min\{r_{M_1}(C_1^*), k\}.$$

As M_1 is vertically $(k+1)$-connected, it follows that $r_{M_1}(C_1^*) \geq k+1$. Thus, $g(M_1^*) \geq k+1$. By Lemma 2.1(i), $r(M_1) = r(M_2)$. We can repeat the previous argument to M_2 and we conclude also that $g(M_2^*) \geq k+1$. By Theorem 4.1, there is a matroid N which is obtained from M_i, for both $i \in \{1,2\}$, by relaxing a sequence of nests (each along an admissible linear subclass of M_i restricted to it) such that

$$\mathcal{B}(M_1) \triangle \mathcal{B}(M_2) = (\mathcal{B}(N) \setminus \mathcal{B}(M_1)) \cup (\mathcal{B}(N) \setminus \mathcal{B}(M_2)).$$

Let Z be a nest of M_i that was relaxed along an admissible linear subclass $\mathcal{L}$ of $M_i|Z$ to originate N. We have that

$$|\mathcal{C}(M_i|Z) \setminus \mathcal{L}| \leq |\mathcal{B}(N) \setminus \mathcal{B}(M_i)| \leq |\mathcal{B}(M_1) \triangle \mathcal{B}(M_2)| \leq k < r(M_i) = |C|,$$

where C is a large circuit of M_i contained in Z. By Lemma 5.1, we have that $\mathcal{L}$ is the set of circuits of $M_i|Z$ that do not meet a series class S. If $e \in S$, then Z is a tip-hyperplane of M_i having tip e. The result follows. $\square$

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