MATROIDS HAVING SMALL CIRCUMFERENCE

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Abstract. In this paper, we conclude the construction of all the matroids having circumference at most five. We use this result to prove a conjecture of Hochstättler and Jackson, in a special case.

1. Introduction

In this paper, we assume familiarity with matroid theory. The notation and terminology used in this article follow Oxley [22]. For a matroid M that is not free, $\mathrm{circ}(M)$ denotes the *circumference* of M, that is, the maximum cardinality of a circuit of M. In recent years, the circumference of a matroid has appeared in some bounds, for example, in an upper bound for the size of a minimally n-connected matroid and in a lower bound for the size of a n-connected matroid having a circuit whose deletion is also n-connected, for $n \in \{2, 3\}$ (see [11, 12, 13]). Using these bounds and results about matroids with small circumference, it is possible to improve some bounds found in the literature. Lemos and Oxley [12] proved that a connected matroid M has a circuit C such that $M \backslash C$ is connected provided $|E(M)| \geq 3r(M)$. This bound is sharp, but it can be improved provided some other conditions are imposed on the matroid M. To improve it, Junior [7] needed to know all the matroids that have circumference at most five. These matroids are constructed in [7] and in this paper. Here, we deal with the most difficult case: the 3-connected matroids having both rank and circumference equal to five.

Dirac proved that a graph G has a Hamiltonian circuit provided $2d(v) \geq |V(G)|$, for every vertex v of G. Clearly, this condition is satisfied by a d-connected graph, when $2d \geq |V(G)|$. As a consequence of Dirac's Theorem, we have that every d-connected graph, for $2d \geq |V(G)|$, has a Hamiltonian circuit. In the submitted version of [4], Hochstättler and Jackson conjectured that the same result would hold for binary matroids. That is, if $2d \geq r(M) + 1$, then a d-connected binary matroid M has a Hamiltonian circuit, which is a circuit with cardinality $r(M) + 1$, unless $r(M)$ and all the circuits of M are even. As the referee provided a counter-example for it, Hochstättler and Jackson made the following conjecture.

Conjecture 1.1. *Let M be a d-connected binary matroid for some integer d where*

$$d \geq \frac{r(M) + 1}{2}.$$

Then M has a circuit of size at least $r(M)$.

With the main result of this paper, we prove this conjecture for $d = 3$. Moreover, we prove a strong result, where we drop the binary hypothesis from the matroid, as suggested by Hochstättler and Jackson, and we list all the matroids satisfying the hypotheses without a Hamiltonian circuit.

Let J_{10} be the matroid whose representation over $GF(2)$ is given by the following matrix.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

This matroid has been considered by Oxley [18], when he characterized the binary matroids with no four-wheel as a minor. The matroid J_{10} is 3-connected and has both rank and circumference equal to five. Now, we describe the circuits of J_{10}. Label the columns of this matrix by $b_1, b_2, b_3, b_4, b_5, a_1, a_2, a_3, a_4, a_5$ respectively. A subset C of $E(J_{10})$ is a circuit of J_{10} if and only if

(1) when $|C \cap \{a_1, a_2, a_3, a_4, a_5\}|$ is even, there is a 2-subset $\{i, j\}$ of $\{1, 2, 3, 4, 5\}$ such that $C = \{a_i, a_j, b_i, b_j\}$, or
(2) when $|C \cap \{a_1, a_2, a_3, a_4, a_5\}|$ is odd, there is $d_i \in \{a_i, b_i\}$, for each $i \in \{1, 2, 3, 4, 5\}$, such that $C = \{d_1, d_2, d_3, d_4, d_5\}$.

Let J_9 be the matroid obtained from J_{10} by deleting b_5. In the main result of this paper, Theorem 1.1, we prove that a 3-connected matroid having rank at most five is Hamiltonian, unless it is isomorphic to $U_{1,1}$, F_7^*, $\mathrm{AG}(3,2)$, J_9, or J_{10}. (The existence of Hamiltonian circuits in a matroid is very important: nice applications of them can be found in [2] and [3].)

Theorem 1.1. *Suppose that M is a 3-connected matroid non-isomorphic to $U_{1,1}$, F_7^*, $\mathrm{AG}(3,2)$, J_9, or J_{10}. If $r(M) \le 5$, then $\mathrm{circ}(M) = r(M) + 1$.*

We say that a matroid M is a *book* when there are 3-connected rank-2 matroids $M_1, \dots, M_n$, for $n \ge 1$, and an element e such that $E(M_i) \cap E(M_j) = \{e\}$, when $i \ne j$, and $M = P_e(M_1, \dots, M_n)$. Junior [7] construct all the matroids with circumference at most five that are not 3-connected. It is not difficult to see that we need to list only these matroids that are connected and simple. Now, we state Junior's theorem [7].

Theorem 1.2. *Suppose that M is a simple connected matroid that is not 3-connected. If $\mathrm{circ}(M) \le 5$, then there are matroids N, B_1 and B_2 such that N is 3-connected, B_1 and B_2 are books, $E(B_1) \cap E(B_2) = \emptyset, E(B_1) \cap E(N) = \{e\}, E(B_2) \cap E(N) = \{f\}$,*

(i) *$r(N) \in \{2, 3\}$ and $M = P_e(N, B_1) \backslash X$, for some $X \subseteq \{e\}$, or*
(ii) *N is isomorphic to F_7^* and $M = P_e(N, B_1) \backslash X$, for some $X \subseteq \{e\}$, or*
(iii) *$r(N) = 2$ and $M = P_f(P_e(N, B_1), B_2) \backslash X$, for some $X \subseteq \{e, f\}$.*

With the next result of Lemos and Oxley [13], we have the complete picture of the circumference for 3-connected matroids because $M(K_{3,r-2})$, for $r \ge 5$, is a 3-connected matroid having circumference six and rank r.

Theorem 1.3. *Suppose that M is a 3-connected matroid. If $r(M) \ge 6$, then $\mathrm{circ}(M) \ge 6$.*

With the construction of the matroids that have circumference at most five, which is completed in this paper, one can calculate the matroid Ramsey numbers $n(s, t)$, for every $s \le 6$. Some of these numbers have been found in [1], [5], [6] and [25]. We do not compute them here, since Lemos and Oxley obtained all of them in [14]. Non-binary matroids with small circumference are also related to matroids having the k-elimination property for circuits, which has been considered in [8] and [20].

2. Know results

We need the following result from Junior [7].

Proposition 2.1. *Suppose that M is a 3-connected matroid non-isomorphic to $U_{1,1}, F_7^*$, or $\mathrm{AG}(3,2)$. If $r(M) \leq 4$, then $\mathrm{circ}(M) = r(M) + 1$. Moreover, if $r(M) \geq 5$, then $\mathrm{circ}(M) \geq 5$.*

For a matroid M, a subset L of $E(M)$ is said to be a *Tutte-line* of M when $(M|L)^*$ is a loopless matroid having rank equal to two. Observe that L has a partition $\{L_1, L_2, \dots, L_k\}$, for some $k \geq 2$, which is called *canonical*, such that a subset C of L is a circuit of M if and only if $C = L \setminus L_i$, for some $i \in \{1, 2 \dots, k\}$. This Tutte-line is said to be *connected* when $k \geq 3$. From Lemos and Oxley [13], we need the following result.

Lemma 2.1. *If L is a Tutte-line of a matroid M and T is a triad of M such that $T \cap L \neq \emptyset$, then T meets an odd number of sets in the canonical partition of L.*

The next two lemmas follow by the elimination axiom for circuits and *orthogonality*, which is the property that a circuit and a cocircuit of a matroid cannot have exactly one common element.

Lemma 2.2. *Suppose that C is a circuit of a matroid M. If T_1 and T_2 are different triads of M such that $T_1 \cap C = T_2 \cap C \neq \emptyset$, then $(T_1 \cup T_2) \setminus C$ is a 2-cocircuit.*

Lemma 2.3. *Suppose that S is a series class of a matroid M. If $S \cup X_1$ and $S \cup X_2$ are circuits of M for different subsets X_1 and X_2 of $E(M) \setminus S$, then $X_1 \cup X_2$ is a dependent set of M.*

3. Some lemmas

In this section, we assume that M is a 3-connected matroid such that

$$r(M) = \mathrm{circ}(M) = 5.$$

We also assume that C is a circuit of M such that $|C| = \mathrm{circ}(M)$. In this section, we prove that M is isomorphic to J_9 or J_{10}. Since C spans a hyperplane of M, the contraction M/C has rank equal to one. Thus, M/C consists of loops and a single parallel class K. Observe that K is a cocircuit of M, being the complement of the hyperplane $\mathrm{cl}_M(C)$; since M is 3-connected, K has at least three elements. Let $\mathcal{D}_M(C)$ be the set of circuits of M/C other than loops. Thus, $\mathcal{D}_M(C)$ is precisely the set of 2-subsets of K; also, since M is 3-connected, no member of $\mathcal{D}_M(C)$ is a circuit of M. The next lemma is fundamental.

Lemma 3.1. *If $D \in \mathcal{D}_M(C)$, then $L = D \cup C$ is a connected Tutte-line having canonical partition $\{D, X_D, Y_D\}$, where $\{X_D, Y_D\}$ is a partition of C such that $|X_D| = 2$ and $|Y_D| = 3$. Moreover, $D \cup X_D$ and $D \cup Y_D$ are circuits of M.*

Proof. It is not difficult to see that L is a connected Tutte-line of M. Suppose that $\{P_0, P_1, \dots, P_k\}$ is the canonical partition of L, for some integer $k \geq 2$. We may suppose that $P_0 = D$ because $D = L \setminus C$. We have

$$5 = \mathrm{circ}(M) \geq |L \setminus P_i|,$$

for every $i \in \{1, \ldots, k\}$. So, $|P_i| \geq 2$. As $\{P_1, \ldots, P_k\}$ is a partition of C, it follows that $k = 2$. Moreover, $\{|P_1|, |P_2|\} = \{2, 3\}$. The result follows. $\square$

As M is 3-connected, it follows that $r(E(K)) \geq 3$. Let

$$\mathcal{X} = \{X : X \text{ is a 3-subset of } E(K) \text{ that is independent in } M\}.$$

Thus, $\mathcal{X} \neq \emptyset$.

Lemma 3.2. *If $X \in \mathcal{X}$, then $M|(C \cup X)$ has at most one 2-cocircuit contained in C.*

Proof. Suppose that C_1^* and C_2^* are different 2-cocircuits of $M|(C \cup X)$ that are contained in C. We have two cases to deal with: either

(1) $|C_1^* \cap C_2^*| = 1$ or
(2) $C_1^* \cap C_2^* = \emptyset$.

In both cases, we prove that there is a series class S of $M|(X \cup C)$ such that $S \cup D_1$ and $S \cup D_2$ are circuits of M, for different 2-subsets D_1 and D_2 of X. By Lemma 2.3, X is dependent in M and we arrive at a contradiction.

In this paragraph, we assume that (1) occurs. So, $C_1^* \cup C_2^*$ is contained in a series class S of $M|(C \cup X)$. As $|S| \geq 3$, it follows that $S = Y_D$, for every 2-subset D of X, because, by Lemma 3.1, X_D and Y_D are the series class of $M|(C \cup D)$ contained in C. Again, by Lemma 3.1, $S \cup D$ is a circuit of M.

Suppose that (2) happens. For every 2-subset D of X, there is an $i \in \{1, 2\}$ such that $X_D = C_i^*$ and $C_{3-i}^* \subset Y_D$, by Lemma 3.1. Thus, there are different 2-subsets D_1 and D_2 of X and $i \in \{1, 2\}$ such that $X_{D_1} = X_{D_2} = C_i^*$. Hence $D_1 \cup S$ and $D_2 \cup S$ are circuits of M, where $S = C_i^*$, in this case. $\square$

Lemma 3.3. *If D is a 2-subset of X, where $X \in \mathcal{X}$, then $X_D \cup (X \setminus D)$ is a triad of $M|(C \cup X)$.*

Proof. Suppose that $X_D \cup (X \setminus D)$ is not a triad of $M|(C \cup X)$. As X_D is a 2-cocircuit of $M|(C \cup D)$, it follows that X_D is a 2-cocircuit of $M|(C \cup X)$. So, for every 2-subset D' of X such that $D \neq D'$,

(1) $X_D = X_{D'}$ or
(2) $X_D \subset Y_{D'}$.

We prove that (2) holds. If (1) occurs, then $D \cup X_D$ and $D' \cup X_D$ are both circuits of M, by Lemma 3.1. As X_D is a series class of $M|(C \cup X)$, it follows that X is dependent in M, by Lemma 2.3. We arrive at a contradicition and (2) happens. By Lemma 3.2, $X_{D'}$ is not a 2-cocircuit of $M|(C \cup X)$. Hence $T^* = X_{D'} \cup (X \setminus D')$ is a triad of $M|(C \cup X)$. By (2), $X_{D'} \subset Y_D$. We arrive at a contradiction, by Lemma 2.1, because T^* meets two sets in the canonical partition of the Tutte-line $C \cup D$. The result follows. $\square$

Suppose that D_1, D_2, D_3 are 2-subsets of a set W. Then, $D_1 \triangle D_2 \triangle D_3 = \emptyset$ if and only if D_1, D_2, D_2 are pairwise different and $|D_1 \cup D_2 \cup D_3| = 3$.

Lemma 3.4. *Suppose that $X \in \mathcal{X}$. If D_1, D_2, D_3 are 2-subsets of X such that $D_1 \triangle D_2 \triangle D_3 = \emptyset$, then $X_{D_1} \triangle X_{D_2} \triangle X_{D_3} = \emptyset$. Moreover, $S = C \setminus (X_{D_1} \cup X_{D_2} \cup X_{D_3})$ is a 2-cocircuit of $M|(C \cup X)$ and $M|(C \cup X)/e$ is isomorphic to F_7^*, for both elements $e \in S$. Moreover, every circuit of $M|(C \cup X)$ meets X is an even number of elements.*

Proof. By Lemma 3.3, for every $i \in \{1,2,3\}$, $T_i^* = X_{D_i} \cup (X \setminus D_i)$ is a triad of $M|(C \cup X)$. When $\{i,j\}$ is a 2-subset of $\{1,2,3\}$, by Lemma 2.1, T_i^* meets all the sets in the canonical partition of the Tutte-line $C \cup D_j$. In particular,

$$X_{D_i} \neq X_{D_j} \text{ and } X_{D_i} \cap X_{D_j} \neq \emptyset. \tag{3.1}$$

By the elimination axiom for circuits, there is a circuit of M contained in

$$[(D_1 \cup X_{D_1}) \cup (D_2 \cup X_{D_2})] \setminus \{e\} = D_3 \cup (X_{D_1} \cup X_{D_2}),$$

where $e \in D_1 \cap D_2$. By Lemma 3.1,

$$X_{D_3} \subseteq X_{D_1} \cup X_{D_2} \text{ or } Y_{D_3} = X_{D_1} \cup X_{D_2}. \tag{3.2}$$

So, by equations 3.1 and 3.2, $X_{D_3} \subseteq X_{D_1} \cup X_{D_2}$, and so

$$X_{D_1} \triangle X_{D_2} \triangle X_{D_3} = \emptyset,$$

and the first part of the lemma follows. By the paragraph prior to the statement of this lemma, $|S| = 2$.

Next, we prove that $M|(C \cup X)$ does not have a circuit that contains X. Suppose that $X \subseteq C' \in \mathcal{C}(M|(C \cup X))$. By orthogonality, $C' \cap T_i^* \neq X \setminus D_i$, for every $i \in \{1,2,3\}$. Thus, C' meets $C \setminus S$ in at least two elements. As $|C'| \leq \mathrm{circ}(M) = 5$, it follows that $|C'| = 5$ and $|C' \cap (C \setminus S)| = 2$. Without loss of generality, we may suppose that $C' = X \cup X_{D_1}$ because X_{D_1}, X_{D_2} and X_{D_3} are all the 2-subsets of $C \setminus S$. We arrive at a contradiction, since $D_1 \cup X_{D_1}$ is a circuit of M which is properly contained in C'. So, C' does not exist. Thus,

$$\mathcal{C}(M|(C \cup X)) = \mathcal{C}(M|(C \cup D_1)) \cup \mathcal{C}(M|(C \cup D_2)) \cup \mathcal{C}(M|(C \cup D_3)).$$

In particular, S is a series class of $M|(C \cup X)$. Moreover, all seven circuits of $M|(C \cup X)/e$ have four elements and the symmetric difference of any two of them is a circuit of M. So, $M|(C \cup X)/e$ is isomorphic to F_7^*. □

We define

$$\mathcal{Y} = \{Y : Y \text{ is a 4-subset of } E(K) \text{ such that every 3-subset of } Y \text{ belongs to } \mathcal{I}(M)\}.$$

Note that this set may be empty. The next lemma is similar to Lemma 3.3.

Lemma 3.5. *If D is a 2-subset of Y, where $Y \in \mathcal{Y}$, then $X_D \cup (Y \setminus D)$ is a cocircuit of $M|(C \cup Y)$.*

Proof. Suppose that $Y \setminus D = \{a, b\}$. For $z \in \{a, b\}$, taking X equal to $D \cup \{z\}$ in Lemma 3.3, we obtain that $T_z^* = X_D \cup \{z\}$ is a triad of $M|(C \cup X)$. Suppose that T_a^* and T_b^* are both triads of $M|(C \cup Y)$. By Lemma 2.2, $Y \setminus D$ is a 2-cocircuit of $M|(C \cup Y)$ because $C \cap T_a^* = C \cap T_b^* = X_D$. We arrive at a contradiction because Y is a cocircuit of $M|(C \cup Y)$. Thus, $X_D \cup (Y \setminus D)$ is a cocircuit of $M|(C \cup Y)$. □

For a set Z, let K_Z be the complete graph having Z as vertex-set. We can view a 2-subset of Z as an edge of K_Z. If D_1, D_2, D_3 are 2-subsets of Z such that $D_1 \triangle D_2 \triangle D_3 = \emptyset$, then D_1, D_2, D_3 are edges of a triangle of K_Z. In the next lemma, it is convinient to regard a triangle of a graph simply as a set of edges rather than as a subgraph.

Lemma 3.6. *Suppose that $Y \in \mathcal{Y}$.*

(i) *If Y is independent in M, then $M|(C \cup Y)$ is isomorphic to J_9.*
(ii) *If Y is a circuit of M, then C contains a non-trivial series class S of $M|(C \cup Y)$ and $M|(C \cup Y)/e$ is isomorphic to* $\mathrm{AG}(3,2)$*, for both elements $e \in S$.*

Proof. Consider the function $\phi : E(K_Y) \to E(K_C)$ such that $\phi(D) = X_D$. This function can be extended to a subset T of $E(K_Y)$ by setting $\phi(T) = \{\phi(D) : D \in T\}$. By Lemma 3.4 and the comments before this lemma, this function takes a triangle of K_Y into a triangle of K_C. We prove that either

(1) there is an injective function $\psi : Y \to C$ such that $\phi(D) = \{\psi(d) : d \in D\}$ or
(2) there is a triangle T' of K_C such that $\phi(T) = T'$, for every triangle T of K_Y.

Suppose that there are different triangles T_1 and T_2 of K_Y such that $\phi(T_1) = \phi(T_2)$. Let T_3 be another triangle of K_Y. As T_3 meets T_1 and T_2 in different edges, it follows that $\phi(T_3)$ meets $\phi(T_1)$ in at least two edges. Thus, $\phi(T_3) = \phi(T_1)$. In this case, we have (2). So, we may suppose that $\phi(T_1) \neq \phi(T_2)$, when T_1 and T_2 are different triangles of K_Y. In this case, it is easy to see that we have (1).

Suppose that (1) happens. In this case, we can label the elements of C by a_1, a_2, a_3, a_4, a_5 and the elements of Y by b_1, b_2, b_3, b_4 such that $\psi(b_i) = a_i$, for every $i \in \{1,2,3,4\}$. We prove that $\mathcal{C}(M|(C \cup Y)) = \mathcal{C}_4 \cup \mathcal{C}_5 \cup \{\{b_1, b_2, b_3, b_4, a_5\}\}$, where

$$\mathcal{C}_4 = \{\{a_i, a_j, b_i, b_j\} : \ \{i,j\} \text{ is a 2-subset of } \{1,2,3,4\}\}$$

and $C' \in \mathcal{C}_5$ if and only if $a_5 \in C' \subseteq C \cup Y, |C' \cap \{a_i, b_i\}| = 1$, for every $i \in \{1,2,3,4\}$, and $|C' \cap \{b_1, b_2, b_3, b_4\}| \in \{0, 2\}$. By Lemma 3.1, $\mathcal{C}_4 \cup \mathcal{C}_5$ is the set of circuits of $M|(C \cup Y)$ that intersects Y in at most two elements.

In this paragraph, we prove that $C' = Y \cup \{a_5\}$, when $C' \in \mathcal{C}(M|(C \cup Y)) \setminus (\mathcal{C}_4 \cup \mathcal{C}_5)$. By Lemma 3.4, every circuit of $M|(C \cup Y)$ intersects Y in an even number of elements. Thus, C' must contain Y because $C' \notin \mathcal{C}_4 \cup \mathcal{C}_5$. There is an $i \in \{1,2,3,4,5\}$ such that $C' \subseteq Y \cup \{a_i\}$, since $\mathrm{circ}(M) = 5$. We prove that $i \notin \{1,2,3,4\}$. Suppose that $i \in \{1,2,3,4\}$. By symmetry, we do not loose generality supposing $i = 1$. There is a circuit C'' of M such that

$$C'' \subseteq (C' \cup \{a_1, a_2, b_1, b_2\}) \setminus \{b_1\} = \{a_1, a_2, b_2, b_3, b_4\}.$$

By Lemma 3.5, $\{b_1, b_2, a_3, a_4\}$ is a cocircuit of $M|(C \cup Y)$. By orthogonality, we have that $b_2 \notin C''$. Hence $C'' \subseteq \{a_1, a_2, b_3, b_4\}$ is a proper subset of the circuit $\{b_3, b_4\} \cup Y_{\{b_3, b_4\}} = \{a_1, a_2, b_3, b_4, a_5\}$. We arrive at a contradiction. Hence $i \notin \{1,2,3,4\}$. Thus, $C' = Y \cup \{a_5\}$.

To conclude that $M|(C \cup Y)$ is isomorphic to J_9, we need to prove only that $Y \cup \{a_5\}$ is a circuit of M. If $Y \cup \{a_5\}$ is not a circuit of M, then, by the previous paragraph, $Y \cup \{a_5\}$ is an independent set of M. As $r(M) = 5$, it follows that $Y \cup \{a_5\}$ is a basis of M. So, there is a circuit C'' of M such that $a_4 \in C'' \subseteq Y \cup \{a_4, a_5\}$. By the previous paragraph, $C'' \in \mathcal{C}_4 \cup \mathcal{C}_5$. Hence $|C'' \cap Y| = 2$ and $C'' \in \mathcal{C}_4$. We arrive at a contradiction and $M|(C \cup Y)$ is isomorphic to J_9.

Now, suppose that (2) holds. It is not difficult to see that $X_D = X_{Y \setminus D}$ for every 2-subsets D of Y. Consider

$$S = \bigcap_D Y_D,$$

where the intersection is taken over all 2-subsets D of Y. So, $|S| = 2$. To conclude that S is a series class of $M|(C \cup Y)$ we need only prove that Y is a circuit of M, since S is a series class of $M|(C \cup (Y \setminus \{y\}))$, by Lemma 3.4, for every $y \in Y$. Suppose that Y is not a circuit of M. First, we prove that $Y \cup \{b\}$ is a circuit of M, for every $b \in C \setminus S$. Choose a 2-subset D of Y such that $b \in X_D$, say $X_D = \{a, b\}$. Hence $D \cup X_D$ and $(Y \setminus D) \cup X_{Y \setminus D} = (Y \setminus D) \cup X_D$ are circuits of M. So, there is a circuit C' of M such that

$$C' \subseteq [(D \cup X_D) \cup ((Y \setminus D) \cup X_D)] \setminus \{a\} = Y \cup \{b\}.$$

Now, we prove that $C' = Y \cup \{b\}$. If $C' \neq Y \cup \{b\}$, then there is $y \in Y \setminus C'$. Hence C' is a circuit of $M|(C \cup (Y \setminus \{y\}))$ such that $C' \cap C = \{b\}$. We arrive at a contradiction by Lemma 3.4. Thus, $C' = Y \cup \{b\}$. Next, we prove that $Y \cup \{e\}$ is a circuit of M, for both elements $e \in S$. Let D' be a 2-subset of Y. Suppose that $Y_{D'} = S \cup \{b\}$. So, there is a circuit C'' of M such that

$$e \in C'' \subseteq [(Y \cup \{b\}) \cup (D' \cup Y_{D'})] \setminus \{b\} = Y \cup S.$$

Observe that $Y \subset C''$ because, for every $y \in Y$, every circuit of $M|(C \cup (Y \setminus \{y\}))$ that meets S must intersect C in three elements, by Lemma 3.4. As $\mathrm{circ}(M) = 5$, it follows that $C'' = Y \cup \{e\}$. In summary, $Y \cup \{f\}$ is a circuit of M, for every $f \in C$. Thus, Y spans $C \cup Y$ in M. We arrive at a contradiction because $r(Y \cup C) = r(C) + 1 = 5$. Hence Y is a circuit of M and S is a series class of $M|(C \cup Y)$. It is not difficult to see that $M|(C \cup Y)/e$ is isomorphic to $\mathrm{AG}(3, 2)$, for both elements $e \in S$. □

Lemma 3.7. *The matroid $M \setminus \mathrm{cl}_M(C)$ is triangle-free.*

Proof. Suppose that T is a triangle of M such that $T \cap \mathrm{cl}_M(C) = \emptyset$. Choose $X \in \mathcal{X}$ such that $|X \cap T| = 2$. Set $Z = X \cup T$. Let D be a 2-subset of X such that $D \neq T \cap X$. By Lemma 3.3, $(X \setminus D) \cup X_D$ is a triad of $M|(C \cup X)$. By orthogonality with T, $C^* = (Z \setminus D) \cup X_D$ is a cocircuit of $M|(C \cup Z)$. Consider the circuit $C' = D \cup Y_D$ of M. By Lemma 3.4, $\mathrm{cl}_{M|(C \cup X)}(C') = C'$. If $T \setminus X \subseteq \mathrm{cl}_{M|(C \cup Z)}(C')$, then there is circuit C'' of M such that $T \setminus X \subseteq C'' \subseteq C' \cup (T \setminus X)$. We arrive at a contradiction, by orthogonality, because $C^* \cap C'' = T \setminus X$. So, $\mathrm{cl}_{M|(C \cup Z)}(C') = C'$. We define $\mathcal{Y}'$ by replacing C by C' in the definition of $\mathcal{Y}$, that is,

$$\mathcal{Y}' = \{Y : Y \text{ is a 4-subset of } E(K') \text{ such that every 3-subset of } Y \text{ belongs to } \mathcal{I}(M)\},$$

where K' denotes the connected component of M/C' having rank equal to one. If

$$(C \cup Z) \setminus C' = X_D \cup (Z \setminus D) = C^*$$

belongs to $\mathcal{Y}'$, then $M|(C' \cup C^*)$ is isomorphic to J_9 or is a series extension of $\mathrm{AG}(3, 2)$, by Lemma 3.6 applied taking C' instead of C. In both cases, we arrive at a contradiction because T is a triangle of $M|(C' \cup C^*)$. So, $C^* \notin \mathcal{Y}'$ and C^* contains a triangle T' of M. We arrive at a contradiction, by Lemma 3.1, because T' is a circuit of $M|(C \cup (Z \setminus D))$. Thus, T does not exist and the result follows. □

Lemma 3.8. $\mathrm{cl}_M(C) = C$.

Proof. Suppose that $g \in \mathrm{cl}_M(C) \setminus C$. Choose $X \in \mathcal{X}$. We prove that $(X \setminus D) \cup X_D$ is a triad of $M|(C \cup X \cup \{g\})$, for every 2-subset D of X. If $(X \setminus D) \cup X_D$ is not a triad of $M|(C \cup X \cup \{g\})$, for some 2-subset D of X, then, by Lemma 3.3, $C^* = (X \setminus D) \cup X_D \cup \{g\}$ is a cocircuit of $M|(C \cup X \cup \{g\})$. As $C' = D \cup Y_D$ is a circuit of M with cardinality five and $C' \cap C^* = \emptyset$, it follows that $M|(C' \cup C^*)$ is isomorphic to J_9 or is a series extention of $\mathrm{AG}(3, 2)$, by Lemmas 3.6 and 3.7 applied to C' instead of C. We arrive at a contradiction because these two matroids do not have a circuit spanning any element which does not belong to it. Hence $(X \setminus D) \cup X_D$ is a triad of $M|(C \cup X \cup \{g\})$, for every 2-subset D of X.

Observe that $S = \cap Y_D$, where the intersection is taken over the 2-subsets D of X, is a series class of $M|(C \cup X)$, by Lemma 3.4. Let B be a basis of C such that $S \subseteq B$. As $g \in \mathrm{cl}_M(B)$, there is a circuit C'' of M such that $g \in C'' \subseteq B \cup \{g\}$.

If D is a 2-subset of X such that $X_D \not\subseteq B$, then $X_D \cap C'' = \emptyset$, by orthogonality with the triad $(X \setminus D) \cup X_D$ of $M|(C \cup X \cup \{g\})$. Hence $C'' = S \cup \{g\}$. Thus, $\mathrm{cl}_M(S) \setminus S = \mathrm{cl}_M(C) \setminus C$.

In this paragraph, we prove that $r(E(M) \setminus \mathrm{cl}_M(C)) \geq 4$. Suppose that $r(E(M) \setminus \mathrm{cl}_M(C)) = 3$. So, $E(M) \setminus \mathrm{cl}_M(C) \subseteq \mathrm{cl}_M(X)$. As S is a series class of $M|(C \cup X)$, it follows that

$$1 + r(M) = r(X \cup (C \setminus S)) + r(S).$$

As S spans $\mathrm{cl}_M(C) \setminus C$ and X spans $E(M) \setminus \mathrm{cl}_M(C)$ in M, we have that

$$1 + r(M) = r((E(M) \setminus \mathrm{cl}_M(C)) \cup (C \setminus S)) + r(S \cup (\mathrm{cl}_M(C) \setminus C)).$$

We arrive at a contradiction because M is 3-connected and $\{(E(M) \setminus \mathrm{cl}_M(C)) \cup (C \setminus S), S \cup (\mathrm{cl}_M(C) \setminus C)\}$ is a partition of $E(M)$. Hence $r(E(M) \setminus \mathrm{cl}_M(C)) \geq 4$.

Suppose that Y is a 4-subset of $E(M) \setminus \mathrm{cl}_M(C)$ and that $r(Y) = 4$. By Lemma 3.6, $M|(C \cup Y)$ is isomorphic to J_9. So, for each 3-subset X of Y, there is S_X, which is the non-trivial series class of $M|(C \cup X)$, such that $S_X \cup \{g\}$ is a circuit of M, by the second paragraph. Hence $S_X \cup S_{X'}$ contains a circuit of M, when $X \neq X'$, which is properly contained in C. We arrive at a contradiction and g does not exist. □

Corollary 3.1. *$E(M) \setminus C$ is a cocircuit of M.*

Proof. By Lemma 3.8, $\mathrm{cl}_M(C) = C$. As $r(M) = r(C) + 1$, it follows that C is a hyperplane of M. Thus, $E(M) \setminus C$ is a cocircuit of M. □

Lemma 3.9. *If $Y \subseteq E(M) \setminus C$ is a 4-circuit of M, then $E(M) = C \cup Y$.*

Proof. Suppose that $Y \neq E(M) \setminus C$. By Lemma 3.6(ii), there is a 2-subset S of C such that S is a series class of $M|(C \cup Y)$.We prove that $r(E(M) \setminus C) > 3$. Suppose that $r(E(M) \setminus C) = 3$. By Lemmas 3.7 and 3.8, we have that $M|(E(M) \setminus C)$ is isomorphic to $U_{3,|E(M) \setminus C|}$. Thus, S is a series class of M because Y spans $E(M) \setminus C$. We arrive at a contradiction because M is 3-connected. Hence $r(E(M) \setminus C) \geq 4$.

Let Z be a 4-element subset of $E(M) \setminus C$ such that $r(Z) = 4$ and $|Z \cap Y| = 3$. Choose a 2-subset D of Z such that $Z \setminus Y \subseteq D$. As $D \cup Y_D$ is a circuit of M having five elements, it follows, by Corollary 3.1, that

$$C^* = (C \cup Y \cup Z) \setminus (D \cup Y_D) = X_D \cup (Y \setminus D)$$

is a cocircuit of $M|(C \cup Y \cup Z)$. By Lemma 3.6(i) applied to $M|(C \cup Z)$, we have that $|X_D \cap S| = 1$. By Lemma 3.6(ii) applied to $M|(C \cup Y)$, for each 2-subset W of $C \setminus S$, there is a partition $\{D_1, D_2\}$ of Y such that $X_{D_1} = X_{D_2} = W$. Choose W and D_1 such that $W \cap X_D = \emptyset$ and $|(Y \setminus D) \cap D_1| = 1$. So,

$$(D_1 \cup X_{D_1}) \cap C^* = (D_1 \cup X_{D_1}) \cap [X_D \cup (Y \setminus D)] = [(Y \setminus D) \cap D_1)] \cup (X_{D_1} \cap X_D) = (Y \setminus D) \cap D_1.$$

We arrive at a contradiction, by orthogonality, because $D_1 \cup X_{D_1}$ is a circuit of M. □

Corollary 3.2. *If $Y \in \mathcal{Y}$, then Y is independent in M.*

Proof. Suppose that Y is a circuit of M. Thus, by Lemma 3.9, $E(M) = C \cup Y$. We arrive at a contradiction to the fact that M is 3-connected because, by Lemma 3.6(ii), M is a non-trivial series extension of $\mathrm{AG}(3, 2)$. □

Lemma 3.10. *Suppose that Z is a 5-subset of $E(M) \setminus C$ such that $Z \setminus \{f\}$ is independent in M, for every $f \in Z$. Then $M|(C \cup Z)$ is isomorphic to J_{10}. In particular, Z is an independent set of M.*

Proof. For each $f \in Z$, by Lemma 3.6(i), there is a function $\psi_f : Z \setminus \{f\} \to C$ such that

(3.3) if $D = \{a, b\}$ is a 2-subset of $Z \setminus \{f\}$, then $X_D = \{\psi_f(a), \psi_f(b)\}$.

When f and g are different elements of Z, then the restrictions of ψ_f and ψ_g to $Z \setminus \{f, g\}$ agree, by equation (3.3). Hence there is a function $\psi : Z \to C$ such that ψ_f is the restriction of ψ to $Z \setminus \{f\}$, for every $f \in Z$.

By Lemma 3.6(i), when $C' \neq Z$ and $C' \subseteq C \cup Z$, then C' is a circuit of $M|(C \cup Z)$ if and only if there is a 2-subset $\{a, b\}$ of Z such that $C' = \{a, b, \psi(a), \psi(b)\}$ or $|C' \cap \{a, \psi(a)\}| = 1$, for every $a \in Z$, and $|C' \cap Z|$ is even. Moreover, these are the circuits of $M|(C \cup Z)$ which have at most four elements in common with Z. Hence $M|(C \cup Z)$ is isomorphic to J_{10} unless Z is a circuit of M. Suppose that Z is a circuit of M. Thus, $Z \cup \{a\}$ is a Tutte-line of M, for every $a \in C$, since $(Z \setminus \{\psi^{-1}(a)\} \cup \{a\}$ is a circuit of M. So, $(Z \setminus \{b\}) \cup \{a\}$ contains a circuit of M, for every $a \in C$ and $b \in Z$. We arrive at a contradiction by Lemma 3.6(i). □

Proposition 3.1. *Suppose that M is a 3-connected matroid. If $r(M) = \operatorname{circ}(M) = 5$, then M is isomorphic to J_9 or J_{10}.*

Proof. From Lemma 3.8, we have $\operatorname{cl}_M(C) = C$. By Lemma 3.7 and Corollary 3.2, every circuit of $M \backslash C$ has five elements because $\operatorname{circ}(M) = 5$. If $|E(M) \setminus C| \leq 5$, then the result follows from Lemmas 3.4, 3.6 and 3.10. So, we may suppose that $|E(M) \setminus C| \geq 6$. By Lemma 3.10, every 5-subset of $E(M) \setminus C$ is independent in M. Hence $E(M) \setminus C$ is independent in M, otherwise it contains a circuit with at least six elements. We have a contradiction because $r(M) = 5$ and the result follows. □

4. The main results

Note that Theorem 1.1 is a consequence of Propositions 2.1 and 3.1. In this section, we prove a result that is strong than Conjecture 1.1 in the case $d = 3$.

Theorem 4.1. *Suppose that $d = 3$. Let M be a d-connected matroid non-isomorphic to $U_{1,1}, F_7^*, \mathrm{AG}(3,2), J_9$, or J_{10}. If*

$$d \geq \frac{r(M) + 1}{2},$$

then M has a circuit of size $r(M) + 1$.

Proof. As $d = 3$, it follows that $r(M) \leq 5$. Thus, the result follows from Theorem 1.1 □

When M is isomorphic to $F_7^*, \mathrm{AG}(3,2), J_9$, or J_{10}, then M has a circuit of size equal to $r(M)$. Moreover, every circuit of M having size equal to $r(M)$ is a circuit-hyperplane of M. To see the relevance of circuits which are also hyperplanes we refer, for example, to [9], [10], [15], [16], [17], [19], [21], [23] and [24].

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