On Mills's conjecture on matroids with many common bases

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Abstract

In this paper, we shall prove a conjecture of Mills: for two $(k+1)$-connected matroids whose symmetric difference between their collections of bases has size at most k, there is a matroid that is obtained from one of these matroids by relaxing n_1 circuit-hyperplanes and from the other by relaxing n_2 circuit-hyperplanes, where n_1 and n_2 are non-negative integers such that $n_1 + n_2 \leq k$.

1 Introduction

In this article, for matroid theory we shall use the notation set by Oxley in [2], which we assume familiar to the reader. We shall prove, up to small modifications, the following conjecture made by Mills in [1].

Conjecture 1. Let k be a positive integer. Suppose M_1 and M_2 are $(k+1)$-connected matroids on a set E where $|E| \geq 2$. If $|\mathcal{B}(M_1) \triangle \mathcal{B}(M_2)| = k$, then there is an integer $j \in [0, k]$ and a matroid N on E that is obtained from M_1 and M_2 by relaxing j and $k - j$ circuit-hyperplanes, respectively.

For $j = 0$ and $i \in \{1, 2\}$, the matroid obtained from M_i by relaxing j circuits-hyperplanes is defined to be M_i.

Observe that, for $r \geq 1$, $U_{r-1,2r-1}$ and $U_{r,2r-1}$ are m-connected matroids for every positive integer m. As these matroids have different rank, it follows that

$$|\mathcal{B}(U_{r-1,2r-1}) \triangle \mathcal{B}(U_{r,2r-1})| = |\mathcal{B}(U_{r-1,2r-1}) \cup \mathcal{B}(U_{r,2r-1})| = 2\binom{2r-1}{r} = f(r).$$

When $k = f(r)$, this pair of matroids is a counter-example to Conjecture 1. We shall prove that these are the only exceptions and that the hypothesis on the size of E can be removed from the conjecture.

In [1], Mills proved this conjecture for $k = 2$ and pointed out that it follows from Truemper's results [3] when $k = 1$. In general, this conjecture follows from a theorem which shall be proved in the next section and guarantees the same conclusion with weaker hypotheses.

2 The main result

For a matroid M, the *girth* of M is defined as

$$\mathrm{g}(M) = \min\{|C| : C \in \mathcal{C}(M)\},$$

where this minimum is taken to be 0, when $\mathcal{C}(M) = \emptyset$. We denote by $\mathcal{CH}(M)$ the collection of circuit-hyperplanes of M. A subset L of $E(M)$ is said to be a *Tutte-line*, when $M|L$ has corank equal to two and no coloops. In [4], Tutte proved that L has a partition, which we call *the canonical partition of L in M*, $\{P_1, P_2, \dots, P_n\}$, for some $n \geq 2$, such that $\mathcal{C}(M|L) = \{L \backslash P_1, L \backslash P_2, \dots, L \backslash P_n\}$. We say that (M, N) is a k-pair, for a non-negative integer k, when N is a matroid having the same ground set as M and $|\mathcal{B}(M) \bigtriangleup \mathcal{B}(N)| \leq k$. Observe that this property is invariant under duality; that is, (M, N) is a k-pair if and only if (M^*, N^*) is a k-pair.

Lemma 1. Let k be a positive integer. If (M_1, M_2) is a k-pair and

$$\max\{\mathrm{g}(M_1), \mathrm{g}(M_1^*)\} \geq k + 1,$$

then $r(M_1) = r(M_2)$.

Proof: Taking the dual, when necessary, we do not lose generality by supposing that $\mathrm{g}(M_1) \geq k + 1$. We shall argue by contradiction. Suppose that $r(M_1) \neq r(M_2)$. If C is a circuit of M_1, then $C \setminus \{e\}$ can be completed to a basis B_e of M_1, for every $e \in C$. Observe that

$$\{B_e : e \in C\} \subseteq \mathcal{B}(M_1) \subseteq \mathcal{B}(M_1) \cup \mathcal{B}(M_2) = \mathcal{B}(M_1) \bigtriangleup \mathcal{B}(M_2),$$

where the last equality holds because M_1 and M_2 do not have any bases in common. As $|C| \geq \mathrm{g}(M_1) \geq k + 1$, it follows that the cardinality of the first set is at least $k + 1$. We arrive at a contradiction, since (M_1, M_2) is a k-pair, by hypothesis. Thus $r(M_1) = r(M_2)$. QED

Lemma 2. Let k be a positive integer. If (M_1, M_2) is a k-pair and $\mathrm{g}(M_1^*) \geq k + 1$, then

$$\mathcal{I}(M_1) \cap \mathcal{C}(M_2) \subseteq \mathcal{B}(M_1).$$

Proof: Suppose that $C \in \mathcal{I}(M_1) \cap \mathcal{C}(M_2)$. We need to prove only that $C \in \mathcal{B}(M_1)$. To conclude this we will show that C spans M_1. Suppose that C does not span M_1. So, there is a cocircuit D of M_1 such that $C \cap D = \emptyset$. However, as D is a cocircuit of M_1/C, for each $d \in D$, there is a basis B'_d of M_1/C such that $B'_d \cap D = \{d\}$. Hence $B'_d \cup C$ is a basis of M_1 for every $d \in D$. As $B'_d \cup C$ is not a basis for M_2, it follows that

$$|\mathcal{B}(M_1) \triangle \mathcal{B}(M_2)| \geq |\{B'_d \cup C : d \in D\}| = |D| \geq \mathrm{g}(M_1^*) \geq k+1.$$

We arrive at a contradiction because (M_1, M_2) is a k-pair. Thus C spans M_1 and the result follows. QED

Lemma 3. Let k be a positive integer. If (M_1, M_2) is a k-pair such that $\min\{\mathrm{g}(M_1), \mathrm{g}(M_1^*)\} \geq k+1$, then

$$\mathcal{I}(M_1) \cap \mathcal{C}(M_2) = \mathcal{B}(M_1) \cap \mathcal{CH}(M_2).$$

Proof: By Lemma 1, we have that $r(M_1) = r(M_2)$. Suppose that $C \in \mathcal{I}(M_1) \cap \mathcal{C}(M_2)$. By Lemma 2, it follows that $|C| = r(M_1)$ and so $|C| = r(M_2)$. In the next paragraph we shall prove that $C \in \mathcal{CH}(M_2)$.

Suppose that $C \notin \mathcal{CH}(M_2)$. Thus $\mathrm{cl}_{M_2}(C) \neq C$, since $|C| = r(M_2)$. Let a be an element belonging to $E(M_2) \setminus C$ spanned by C in M_2. Observe that $C \cup \{a\}$ is a Tutte-line of M_2. Let $\{P_1, P_2, \ldots, P_n\}$ be the canonical partition of $C \cup \{a\}$ in M_2, for some $n \geq 2$. Set $C_i = (C \cup \{a\}) \setminus P_i$, for $i \in \{1, 2, \ldots, n\}$. There is just one circuit C_0 of M_1 contained in $C \cup \{a\}$ because C is a basis of M_1, by Lemma 2. Let

$$\mathcal{B}' = \{C_i : i \in \{1, 2, \ldots, n\} \text{ and } C_0 \not\subseteq C_i\}.$$

So, $\mathcal{B}' \subseteq \mathcal{I}(M_1) \cap \mathcal{C}(M_2)$. Thus, by Lemma 2, $\mathcal{B}' \subseteq \mathcal{B}(M_1)$. So,

$$\mathcal{B}' \subseteq \mathcal{B}(M_1) \triangle \mathcal{B}(M_2).$$

Hence there is an integer l such that $|\mathcal{B}'| = l \leq k$, since (M_1, M_2) is a k-pair. Say, $\mathcal{B}' = \{C_1, \ldots, C_l\}$. As $|C_i| = |C| = r(M_1)$, for every $i \in \{1, \ldots, l\}$, it follows that $P_i = \{a_i\}$, for some $a_i \in E(M_2)$. Hence $\{a_1, \ldots, a_l\} \subseteq C_0$. If $j > l$, then $C_0 \subseteq C_j$ and $C_0 \cap P_j = \emptyset$. Thus $C_0 = \{a_1, \ldots, a_l\}$. We arrive at a contradiction because

$$k \geq l = |C_0| \geq \mathrm{g}(M_1) \geq k+1.$$

So, $C \in \mathcal{CH}(M_2)$. That is,

$$\mathcal{I}(M_1) \cap \mathcal{C}(M_2) \subseteq \mathcal{CH}(M_2).$$

From the previous paragraph and Lemma 2, we have that

$$\mathcal{I}(M_1) \cap \mathcal{C}(M_2) \subseteq \mathcal{B}(M_1) \cap \mathcal{CH}(M_2).$$

As every element in $\mathcal{B}(M_1) \cap \mathcal{CH}(M_2)$ belongs to $\mathcal{I}(M_1) \cap \mathcal{C}(M_2)$, then we have equality in this inclusion and the result follows. QED

Recall that a matroid is *paving* if it has no circuits of size less than $r(M)$.

Lemma 4. Let k be a positive integer. If (M_1, M_2) is a k-pair and

$$\min\{g(M_1), g(M_1^*)\} \geq k+1,$$

then either

(a) $\min\{g(M_2), g(M_2^*)\} \geq k+1$, or

(b) M_1 is a uniform matroid and M_2 is a paving matroid having the same rank as M_1, which is equal to k or $|E(M_1)| - k$. Moreover,

$$\{C : C \in \mathcal{C}(M_2) \text{ and } |C| = r(M_2)\} = \mathcal{CH}(M_2).$$

Proof: We shall assume that (a) does not hold and we shall prove (b). Observe that the hypotheses and the conclusions of this lemma are invariant under duality. Thus we can suppose that $g(M_2) \leq k$. Let C be a circuit of M_2 such that $|C| = g(M_2)$. Observe that $C \in \mathcal{I}(M_1)$, since $g(M_1) \geq k+1$. Hence, by Lemma 3, $C \in \mathcal{B}(M_1) \cap \mathcal{CH}(M_2)$. So,

$$r(M_1) = |C| = g(M_2) \leq k \leq g(M_1) - 1 \leq c(M_1) - 1 \leq r(M_1),$$

where $c(M_1)$ denotes the circumference of M_1; that is, the size of the largest circuit of M_1. Thus, all inequalities in the previous display are actually equalities. Since

$$g(M_1) = c(M_1) = r(M_1) + 1,$$

we have that M_1 is a uniform matroid. Moreover, as $k = g(M_2) = r(M_1)$, it follows from Lemma 1 that M_2 is a paving matroid and $r(M_2) = r(M_1) = k$.

Now suppose that $C \in \mathcal{C}(M_2)$ and $|C| = r(M_2)$. Since $r(M_2) = g(M_2) = k$, the set C is independent in M_1 as $g(M_1) \geq k+1$. It follows from Lemma 3 that $C \in \mathcal{C}H(M_2)$. QED

Lemma 5. Let k be a positive integer. If (M_1, M_2) is a k-pair and

$$\min\{g(M_1), g(M_1^*)\} \geq k+1,$$

then

$$\mathcal{B}(M_i) \setminus \mathcal{B}(M_{3-i}) = \mathcal{CH}(M_{3-i}) \setminus \mathcal{CH}(M_i),$$

for every $i \in \{1, 2\}$.

Proof: When M_1 and M_2 are as in (b) of Lemma 4, then Lemma 5 follows because

$$\mathcal{CH}(M_1) = \mathcal{B}(M_2) \setminus \mathcal{B}(M_1) = \emptyset \text{ and}$$

$$\mathcal{CH}(M_2) = \{C : C \in \mathcal{C}(M_2) \text{ and } |C| = r(M_2)\} = \mathcal{B}(M_1) \setminus \mathcal{B}(M_2).$$

So we may suppose that $\min\{g(M_2), g(M_2^*)\} \geq k+1$. Hence, by symmetry, it suffices to prove this result for $i = 1$.

Let $B \in \mathcal{B}(M_1) \setminus \mathcal{B}(M_2)$. So, there is a circuit C of M_2 such that $C \subseteq B$. As $C \in \mathcal{I}(M_1) \cap \mathcal{C}(M_2)$, it follows, by Lemma 3, that $C \in \mathcal{B}(M_1) \cap \mathcal{CH}(M_2)$. That is, $\mathcal{B}(M_1) \setminus \mathcal{B}(M_2) \subseteq \mathcal{CH}(M_2) \setminus \mathcal{CH}(M_1)$.

Let $C \in \mathcal{CH}(M_2) \setminus \mathcal{CH}(M_1)$. We shall prove that $C \in \mathcal{I}(M_1)$. Suppose that $C \notin \mathcal{I}(M_1)$. Let C' be a circuit of M_1 such that $C' \subseteq C$. Hence $C' = C$, otherwise $C' \in \mathcal{I}(M_2)$ and, by Lemma 3 applied to the k-pair (M_2, M_1), $C' \in \mathcal{B}(M_2)$. Since $C \in \mathcal{CH}(M_2) \setminus \mathcal{CH}(M_1)$, and Lemma 1 implies that $r(M_1) = r(M_2)$, it follows that there is an $a \in E(M_1) \setminus C$ such that C spans a in M_1. Let $\{P_1, P_2, \ldots, P_n\}$ for some $n \geq 2$ be the canonical partition of the Tutte-line $C \cup \{a\}$ in M_1. We may suppose that $P_1 = \{a\}$. Observe that $C_i = (C \cup \{a\}) \setminus P_i$ is independent in M_2, for every $i \in \{2, \ldots, n\}$, because C does not span a in M_2. By Lemma 3 applied to the k-pair (M_2, M_1), it follows that C_i is a circuit-hyperplane of M_1 and a basis of M_2, for $i \in \{2, \ldots, n\}$. So $|C_i| = |C|$ for every $i \in \{1, 2, \ldots, n\}$. In particular, $|P_i| = 1$, for every $i \in \{1, 2, \ldots, n\}$. Thus $n = |C| + 1 \geq \mathrm{g}(M_2) + 1 \geq k + 2$. We arrive at a contradiction to the assumption that (M_1, M_2) is a k-pair because

$$\{C_2, \ldots, C_n\} \subseteq \mathcal{B}(M_2) \setminus \mathcal{B}(M_1).$$

As a result of this contradiction, we conclude that $C \in \mathcal{I}(M_1)$. Moreover, as Lemma 1 implies that $r(M_1) = r(M_2) = |C|$, we have that $C \in \mathcal{B}(M_1)$. Thus $\mathcal{CH}(M_2) \setminus \mathcal{CH}(M_1) \subseteq \mathcal{B}(M_1) \setminus \mathcal{B}(M_2)$, and the result follows. QED

Theorem 1. Let k be a positive integer. If (M_1, M_2) is a k-pair and

$$\min\{\mathrm{g}(M_1), \mathrm{g}(M_1^*)\} \geq k + 1,$$

then there is a matroid N which is obtained from M_1 and M_2 by relaxing n_1 and n_2 circuit-hyperplanes, respectively, where n_1 and n_2 are non-negative integers such that

$$n_1 + n_2 = |\mathcal{B}(M_1) \triangle \mathcal{B}(M_2)| \leq k.$$

Proof: By Lemma 5, we have that

$$\mathcal{B}(M_i) \setminus \mathcal{B}(M_{3-i}) = \mathcal{CH}(M_{3-i}) \setminus \mathcal{CH}(M_i), \tag{$*$}$$

for every $i \in \{1, 2\}$. For $i \in \{1, 2\}$, suppose that

$$n_i = |\mathcal{CH}(M_i) \setminus \mathcal{CH}(M_{3-i})|.$$

Let N_i be the matroid obtained from M_i by relaxing the circuit-hyperplanes belonging to $\mathcal{CH}(M_i) \setminus \mathcal{CH}(M_{3-i})$. By $(*)$, we have that $N_1 = N_2$; call this matroid N. Again, by $(*)$, we obtain

$$n_1 + n_2 = |\mathcal{CH}(M_1) \triangle \mathcal{CH}(M_2)| = |\mathcal{B}(M_1) \triangle \mathcal{B}(M_2)| \leq k,$$

where the last inequality follows because (M_1, M_2) is a k-pair. Thus, the result follows. QED

We say that a matroid M is *square*, when

$$|E(M)| - 2r(M) \in \{-1, 0, 1\}.$$

The next result follows from Corollary 8.1.8 of [2]:

Lemma 6. If M is a $(k+1)$-connected matroid and $|E(M)| < 2k$, then M is a square uniform matroid. QED

Using the previous lemma, we can prove the next result, which removes the bound on the size of E from Conjecture 1 and excludes the few counter-examples to it.

Theorem 2. For some positive integer k, suppose that M_1 and M_2 are $(k+1)$-connected matroids having the same ground set. If $|\mathcal{B}(M_1) \triangle \mathcal{B}(M_2)| = k$, then either

(a) there is an integer $j \in [0, k]$ and a matroid N on E that is obtained from M_1 and M_2 by relaxing j and $k - j$ circuits-hyperplanes, respectively; or

(b) there is a positive integer r such that

$$2\binom{2r-1}{r} = k$$

and $M_1 = M_2^*$ is a square uniform matroid having $2r - 1$ elements.

Proof: When $|E(M)| \geq 2k$, then $\min\{\mathrm{g}(M_1), \mathrm{g}(M_1^*)\} \geq k + 1$ and the result follows from Theorem 1. So we may suppose that $|E(M)| \leq 2k-1$. Then Lemma 6 implies that M_1 and M_2 are square uniform matroids. As $E(M_1) = E(M_2)$ and $M_1 \neq M_2$, it follows that $M_1 = M_2^*$ and $|E(M_1)| = 2r-1$, for some positive integer r. Observe that

$$2\binom{2r-1}{r} = k$$

because $r(M_1) \neq r(M_2)$ and so $\mathcal{B}(M_1) \triangle \mathcal{B}(M_2) = \mathcal{B}(M_1) \cup \mathcal{B}(M_2)$. QED

Acknowledgments

The author was partially supported by CNPq, FINEP and PRONEX 107/97. The author also thanks the referees for the valuable suggestions that improved this article.

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