ON THE 3–CONNECTED MATROIDS THAT ARE MINIMAL HAVING A FIXED RESTRICTION

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Abstract. Let N be a restriction of a 3-connected matroid M and let M' be a 3-connected minor of M that is minimal having N as a restriction. This paper gives a best-possible upper bound on $|E(M') - E(N)|$.

1. Introduction

If X is a subset of a 3-connected matroid M, what can be said about the size of a minimal 3-connected minor M' of M that not only includes X in its ground set but also maintains the matroid structure on X? For instance, if X is a circuit, a basis, or an independent set of M, then X is a circuit, a basis, or an independent set, respectively, of M'. The purpose of this paper is to answer this question. More specifically, we solve the following:

1.1. Problem. *Let N be a restriction of a 3-connected matroid M and let M' be a 3-connected minor of M that is minimal having N as a restriction. Give a sharp upper bound on $|E(M') - E(N)|$.*

The natural modification of this problem in which "restriction" is replaced by "minor" seems much more difficult, although we do hope to return to it in future work. For this modified problem, Truemper [12] proved, under the additional constraints that N is 3-connected but different from M', that $|E(M') - E(N)| \leq 3$. In certain natural cases, including those raised above, when N is a circuit, a basis, or an independent set of M, the modified and original problems coincide.

Let M be a matroid and A be a subset of $E(M)$. We define $\lambda_1(A, M)$ to be the number of connected components of $M|A$. Now $M|A$ can be constructed from a collection $\Lambda_2(A, M)$ of 3–connected matroids by using the operations of direct sum and 2-sum. It follows from results of Cunningham and Edmonds (see Cunningham [3]) that $\Lambda_2(A, M)$ is unique up to isomorphism. We denote by $\lambda_2(A, M)$ the number of matroids in $\Lambda_2(A, M)$ that are not isomorphic to $U_{1,3}$, the three-element cocircuit.

The next theorem, the main result of [6], solves Problem 1.1 in the case when N spans M.

1.2. Theorem. *Let M be a 3–connected matroid other than $U_{1,3}$ and let A be a non-empty spanning subset of $E(M)$. If M has no proper 3–connected minor M' such that $M'|A = M|A$, then*

$$|E(M)| \leq |A| + \lambda_1(A, M) + \lambda_2(A, M) - 2,$$

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unless A is a circuit of M of size at least four, in which case,

$$|E(M)| \leq 2|A| - 2.$$

It is also shown in [6] that Theorem 1.2 is best-possible. Indeed, examples are given that attain the bound for all A such that $M|A$ is simple but not free.

In this paper, we shall prove several results. We solve Problem 1.1 when N is a free matroid by proving the following result.

1.3. Theorem. *Let M be a 3–connected matroid and let A be an independent set of M. If M has no proper 3–connected minor M' in which A is independent, then $|E(M)| \leq 3|A| - 1$.*

Note that this theorem is best-possible. To see this, suppose that $n \geq 3$ and let $K''_{3,n}$ be the graph obtained from $K_{3,n}$ by joining a degree-n vertex v_1 of the latter to the two vertices, v_2 and v_3, to which it is non-adjacent. Then it is not difficult to check that equality is attained in the theorem if we take M to be $M^*(K''_{3,n})$ and A to be any set consisting of all but one of the edges meeting v_1. When A is both independent and spanning, the bound in Theorem 1.3 can be improved by 3 as will be shown in [7].

If A is a subset of the ground set of a 3-connected matroid M, then (M, A) is a *minimal pair* if M has no proper 3–connected minor M' such that $M'|A = M|A$. Thus, in both Theorems 1.2 and 1.3, (M, A) is a minimal pair. Our second theorem shows that the bound in Theorem 1.2 also holds when A is non-spanning, provided A has no small cocircuits. More specifically:

1.4. Theorem. *Let (M, A) be a minimal pair such that $M|A$ has no coloops, every series class of $M|A$ has rank at most two, and $M \not\cong U_{1,3}$. Then*

$$|E(M)| \leq |A| + \lambda_1(A, M) + \lambda_2(A, M) - 2.$$

When A is arbitrary, we shall describe examples in Section 3 to show that $|E(M)|$ can exceed the bound in the last result by an arbitrarily large number. These examples prompt the introduction of another function, $\lambda_3(A, M)$, which is defined by

$$\lambda_3(A, M) = \sum_S (|S| - 3),$$

where the sum is taken over all series classes S of $M|A$ with at least four elements. The following theorem is one of the two main results of the paper.

1.5. Theorem. *If (M, A) is a minimal pair, then*

$$|E(M)| \leq |A| + \lambda_1(A, M) + \lambda_2(A, M) + \lambda_3(A, M) - \alpha(A, M),$$

where

$$\alpha(A, M) = \begin{cases} 0, & \text{when } A \text{ is a circuit of } M \\ 1, & \text{when } A \text{ is not a circuit} \end{cases}$$

Note that Theorem 1.3 will follow immediately from the last theorem for, when A is independent, $\lambda_1(A, M) = \lambda_2(A, M) = |A|$, $\lambda_3(A, M) = 0$, and $\alpha(A, M) = 1$. Although the function $\lambda_3(A, M)$ may look strange, it is indispensable, even for cographic matroids. For example, for $n \geq 3$, let M be $M^*(K''_{3,n})$ and let A be the set of edges meeting the degree-$(n+2)$ vertex of $K''_{3,n}$. Then $M|A$ is a circuit and it is straightforward to check that equality is attained in Theorem 1.5. In Section 3, we shall provide two families of extremal examples for Theorem 1.5 in each of which $M|A$ can be chosen to have many series classes. Prior to that, Section 2 introduces notation, terminology, and some important known results that will be needed. Section 4 develops the properties of minimal pairs that will be used in the proofs of the main results, and these proofs will be given in Section 5.

The essential difference between the bounds in Theorems 1.2 and 1.5 is the presence of $\lambda_3(A, M)$ in the latter. But the former assumes that A spans M. Indeed, if A is non-spanning and (M, A) is a minimal pair, then, for a basis X of M/A, clearly $(M, A \cup X)$ is a minimal pair in which $A \cup X$ is spanning. Hence Theorem 1.2 can be applied to this minimal pair. This crude technique yields the bound

$$|E(M)| \leq |A| + \lambda_1(A, M) + \lambda_2(A, M) - 2 + 3(r(M) - r(A)).$$

In the next theorem, our second main result, we shall show that this bound can be sharpened.

1.6. Theorem. *If (M, A) is a minimal pair, then*

$$|E(M)| \leq |A| + \lambda_1(A, M) + \lambda_2(A, M) + r(M) - r(A) - \alpha(A, M),$$

where

$$\alpha(A, M) = \begin{cases} 0, & \text{when } A \text{ is a circuit of } M \\ 1, & \text{when } A \text{ is not a circuit} \end{cases}$$

If M is an n-spoked wheel with $n \geq 4$ and A is its rim, then (M, A) is a minimal pair for which equality is attained in the last theorem. However, for this example, the difference between the bound in Theorem 1.5 and $|E(M)|$ is $n - 4$.

2. Preliminaries

In this section, we note a number of results that will be used in the proofs of the main theorems. We shall follow Oxley [9] for basic notation and terminology. We shall require some additional terminology and results relating to 3-connected matroids. Tutte [13] calls an element e of a 3–connected matroid M *essential* if neither the deletion $M\backslash e$ nor the contraction M/e remains 3–connected. Tutte showed that every essential element in a 3-connected matroid is in a triangle or a triad. Indeed, triangles and triads appear constantly in the study of 3-connected matroids, and a very useful concept in this study is that of a chain of triangles and triads. Let $T_1, T_2, \ldots, T_k$ be a non-empty sequence of sets each of which is a triangle or a triad of a matroid M such that, for all i in $\{1, 2, \ldots, k-1\}$,

(i) exactly one of T_i and T_{i+1} is a triangle;
(ii) $|T_i \cap T_{i+1}| = 2$; and
(iii) $(T_{i+1} - T_i) \cap (T_1 \cup T_2 \cup \ldots \cup T_i)$ is empty.

Then we call $T_1, T_2, \ldots, T_k$ a *chain* of M of *length* k with *links* $T_1, T_2, \ldots, T_k$. By extending the proof of Tutte's Wheels and Whirls Theorem [13], Oxley and Wu [11] showed that maximal such chains have non-essential elements at both ends.

2.1. Lemma. *Let $T_1, T_2, \ldots, T_k$ be a chain in a 3-connected matroid M. Then M has $k + 2$ distinct elements $e_1, e_2, \ldots, e_{k+2}$ such that $T_i = \{e_i, e_{i+1}, e_{i+2}\}$ for all i. Suppose that $|E(M)| \geq 4$ and M is not a wheel or a whirl. If the chain $T_1, T_2, \ldots, T_k$ is maximal, then the elements can be labelled so that neither e_1 nor e_{k+2} is essential.*

Let $\{e_1, e_2, e_3\}, \{e_2, e_3, e_4\}, \ldots, \{e_k, e_{k+1}, e_{k+2}\}$ be a chain for some $k \geq 3$. Partition $\{e_1, e_2, \ldots, e_{k+2}\}$ into X_e, those e_i for which i is even, and X_o, those e_i for which i is odd. If $\{e_1, e_2, e_3\}$ is a triad, then X_o is the *rim* of the chain and X_e is its set of *spokes*. If $\{e_1, e_2, e_3\}$ is a triangle, then X_e is its rim and X_o is its set of spokes.

A maximal chain in a 3-connected matroid other than a wheel or whirl is called a *fan*. A fan has *type*-1, *type*-2, or *type*-3 if its first and last links consist of, respectively, two triangles, two triads, or one triangle and one triad. Oxley and Wu [11] proved the following result, which will be very useful here.

2.2. Theorem. *Let M be a 3-connected matroid which is not a wheel or a whirl. Suppose that e is an essential element of M. Then e is in a fan. Moreover, this fan is unique unless*

(a) *every fan containing e consists of a single triangle and any two such triangles meet in $\{e\}$;*
(b) *every fan containing e consists of a single triad and any two such triads meet in $\{e\}$;*
(c) *e is in exactly three fans; these three fans are of the same type, each has five elements, together they contain a total of six elements; and, depending on whether these fans are of type-1 or type-2, the restriction or contraction, respectively, of M to this set of six elements is isomorphic to $M(K_4)$.*

The following basic result of Bixby [1] will be used repeatedly throughout the paper.

2.3. Lemma. *Let e be an element of a 3–connected matroid M. Then either $M\backslash e$ or M/e has no non-minimal 2-separations. Moreover, in the first case, the cosimplification of $M\backslash e$ is 3-connected, while in the second case, the simplification of M/e is 3-connected.*

The elementary proof of the next lemma is omitted.

2.4. Lemma. *If $\{Z, W\}$ is a k-separation of a matroid M and $r(Z) = |Z|$, then every k-element subset of Z contains a cocircuit of M.*

The next lemma is a straightforward consequence of *orthogonality*, the property of a matroid that a circuit and a cocircuit cannot have exactly one common element.

2.5. Lemma. *If D is a cocircuit of a matroid M such that $D \cap (\mathrm{cl}(A) - A) \neq \emptyset$, then $D \cap A \neq \emptyset$. In particular, $|D \cap \mathrm{cl}(A)| \geq 2$.*

The following lemma is proved in [5, (2.12)].

2.6. Lemma. *Let e and f be distinct elements of a 3-connected matroid M such that $M\backslash e$ is not 3-connected. If M has triangles T and T' containing e and f respectively such that $|T \cap T'| = 1$ and $T \cup f$ is a cocircuit of M, then e is in a triad of M.*

The last lemma was used in the proof of the following theorem, the main result of [5]

2.7. Theorem. *Let C^* be a cocircuit of a 3-connected matroid M and suppose that, for every element e of C^*, the contraction M/e is not 3-connected. Then M has triangles T_1 and T_2 such that $T_1 \cap C^*$ and $T_2 \cap C^*$ are distinct and non-empty.*

Next we note a useful property of chains of triangles and triads [11, Lemma 3.4].

2.8. Lemma. *Let e_1, e_2, e_3, e_4, e_5 be distinct elements of a 3-connected matroid M that is not isomorphic to $M(K_4)$. Suppose that $\{e_1, e_2, e_3\}$ and $\{e_3, e_4, e_5\}$ are triangles and $\{e_2, e_3, e_4\}$ is a triad of M. Then these two triangles and this one triad are the only triangles and triads of M containing e_3.*

3. Extremal examples

Some examples were given in the introduction showing that equality can be attained in three of the theorems. In this section, we describe families of examples that attain equality in Theorems 1.5 and 1.3.

The examples given here will use the operation of generalized parallel connection [2]. Let the intersection of the ground sets of the matroids M and $M(K_4)$ be Δ, which is a triangle in both matroids. The *generalized parallel connection of $M(K_4)$ and M across Δ* is the matroid $P_\Delta(M(K_4), M)$ whose ground set is the union of the ground set of the two matroids and whose flats are the subsets X of the ground set so that $X \cap E(M(K_4))$ is a flat of $M(K_4)$ and $X \cap E(M)$ is a flat of M. If the elements of Δ are deleted from $P_\Delta(M(K_4), M)$, we obtain the same matroid that we would get by performing a $\Delta - Y$ -exchange on M across Δ.

Let n be an integer exceeding one. An *n-raft* [4] is a matroid of rank $2n-2$ whose ground set is the union of n disjoint triangles such that, for all $k < n$, the union of every set of k of these triangles has rank $2k$. One example of an n-raft is the matroid $M^*(K_{3,n})$. Another is the matroid that is obtained from the direct sum of two n-element circuits $\{x_1, x_2, \dots, x_n\}$ and $\{z_1, z_2, \dots, z_n\}$ by, for each i, freely adding a new element y_i on the line joining x_i and z_i.

We construct our family of examples by beginning with an n-raft N for some $n \geq 3$. Let the distinguished triangles of the raft be $T_1, T_2, \dots, T_n$ where $T_i = \{x_i, y_i, z_i\}$ for all i, and assume that the raft has the additional property that $\{z_1, z_2, \dots, z_n\}$ is a circuit. By repeated generalized parallel connections, attach exactly k distinct copies of $M(K_4)$ across each T_i in N. Let the resulting matroid be M_1. Now, for each i, delete z_i from M_1. Take M to be the dual of the resulting matroid. In each copy of $M(K_4)$ that was attached across some T_i, pick the *opposite element* to z_i, that is, the element that is not in a triangle with z_i. Do this for all i and let A consist of these nk elements together with the $2n$ elements of N that remain in M.

We assert that (M, A) is a minimal pair. First note that N is certainly connected. Moreover, if $\{X, Y\}$ is a 2-separation of N, then neither X nor Y spans N. It is easy to see that each of X and Y may be assumed to be a union of distinguished triangles and from this we obtain a contradiction to the fact that $\{X, Y\}$ is a 2-separation. Thus N is 3-connected. As $M(K_4)$ is also 3-connected, so too is M_1 [10]. Moreover, it is clear that M_1/z_1 does not simplify to a 3-connected matroid. Thus the cosimplification of $M_1\backslash z_1$ is 3-connected. This cosimplification is equal to $M_1\backslash z_1$ since z_1 is in no triads of M_1. By a similar argument, we obtain that $M_1\backslash z_1, z_2$ is 3-connected and, repeating this argument, we eventually obtain that $M_1\backslash z_1, z_2, \dots, z_n$ is 3-connected. But the last matroid is M^*, hence M is 3-connected. Let M' be a minimal 3-connected minor of M such that $M|A = M'|A$. Now, for each element e of M^* that is not in A, there is another element f of $E(M)-A$ such that e and f are both in the same attached $M(K_4)$ in M_1. Moreover, e is in a triangle of M^* with two elements of A so M^*/e has two elements of A in parallel. Thus no 3-connected minor of M^*/e contains all the elements of A. Since

$M^*\backslash e$ has a 2-cocircuit that contains f and some element of A, a 3-connected minor of $M^*\backslash e$ using all the elements of A must be a 3-connected minor of $M^*\backslash e/f$. Since f is also in a triangle of M^* with two elements of A, we conclude no 3-connected minor of $M^*\backslash e$ contains all the elements of A. Thus (M, A) is indeed a minimal pair.

Now $|E(M)| = 3nk + 2n$ and $|A| = nk + 2n$. To find $M|A$, observe that this matroid is the dual of $M^*.A$. But, in the last matroid, there is a parallel class that contains $T_i - z_i$ together with all the elements of A from copies of $M(K_4)$ attached across T_i. Moreover, by considering the n-raft N, it is not difficult to check that the simplification of $M^*.A$ is a circuit. Thus $M|A$ is the cycle matroid of the graph obtained by joining two vertices by n internally disjoint paths each of length $k+2$. Therefore $\lambda_1(A, M) = 1$, $\lambda_2(A, M) = (k+1)n$, $\lambda_3(A, M) = (k-1)n$, and $\alpha(A, M) = 1$. It follows easily that equality holds in Theorem 1.5. One can also easily check that the bound in Theorem 1.6 exceeds $|E(M)|$ by exactly one in this case.

To close this section, we observe that $U_{2,6}$ is a 2-raft. If we perform the above construction beginning with this matroid, we obtain a matroid M for which $M|A$ is independent and equality is attained in Theorem 1.3.

4. Minimal pairs

The purpose of this section is to prove numerous properties of minimal pairs that will be used in the proofs of the main results. The latter will be given in Section 5. We begin with a lemma that gathers together eleven such properties.

Figure 1.

4.1. Lemma. *A minimal pair (M, A) has the following properties:*

(i) *Suppose $e \in E(M) - \text{cl}(A)$. Then $M \backslash e$ has no non-minimal 2-separations and the cosimplification of this matroid is 3-connected. Moreover, the simplification of M/e is not 3-connected.*

(ii) *If $e \in E(M) - \text{cl}(A)$, then e is essential and belongs to a triad of M.*

(iii) *If A is not a spanning set of M, then $|E(M)| > 6$ and $|A| \geq 3$.*

(iv) *If T is a triangle and T^* is a triad of M such that $T \neq T^*$ and $T \cap T^* \neq \emptyset$, then $T^* - T = \{a\}$ for some a in $\text{cl}(A)$.*

(v) *If T^* is a triad of M such that $|T^* - \text{cl}(A)| \geq 2$, then $T^* \cap A \neq \emptyset$.*

(vi) *Let T_1 and T_2 be distinct subsets of $E(M)$ such that both are triangles or both are triads of M. If $(T_1 \cup T_2) - \text{cl}(A) \neq \emptyset$, then $|T_1 \cap T_2| \leq 1$.*

(vii) *If T is a triangle of M such that $\{f_1, f_2\} \subseteq T - \text{cl}(A)$, then, for each i in $\{1, 2\}$, there is a triad of M that meets $\{f_1, f_2\}$ in $\{f_i\}$.*

(viii) *If $\{e_1, e_2, e_3\}$ is a triangle T of M and $T \subseteq E(M) - \text{cl}(A)$, then, for each i in $\{1, 2, 3\}$, there is a unique triad T_i^* of M such that $T_i^* \cap T = T - e_i$. Moreover,*

(a) *for each i, the unique element a_i of $T_i^* - T$ is in A, and $\{a_1, a_2, a_3\}$ is a triad of M;*

(b) *$M^* = P_\Delta(M(K_4), M^* \backslash T)$ where $\Delta = \{a_1, a_2, a_3\}$ and $M(K_4)$ is labelled as in Figure 1, and M/T is 3-connected;*

(c) *$\{a_1, a_2, a_3\}$ is contained in a series class of $(M/T)|A$, and $(M/T)|(A - a_i) = M|(A - a_i)$.*

(ix) *Let $\mathcal{T}$ be the set of triangles T of M that meet both $\text{cl}(A)$ and $E(M) - \text{cl}(A)$. For every T in $\mathcal{T}$, there is a unique element a_T of $T \cap \text{cl}(A)$ and there is a chain T_{1T}^*, T, T_{2T}^* whose rim A_T is contained in $\text{cl}(A)$. Moreover, if T and T' are different elements of $\mathcal{T}$, then*

(a) *$A_T \neq A_{T'}$ and $a_T \neq a_{T'}$;*

(b) *$|(A_T \cup A_{T'}) \cap A| \geq 3$ with equality only when (1) $A_T \cup A_{T'}$ is contained in a series class of $M|\text{cl}(A)$; and (2) $T_{jT}^* = T_{iT'}^*$ for some i and j, or $A_T - T = A_{T'} - T'$.*

(x) *If $r(A) \leq 3$ and A is not spanning, then M is isomorphic to the rank-four wheel having $\text{cl}(A)$ as its rim.*

(xi) *If $e \in E(M) - \text{cl}(A)$ such that M has no triangle that contains e but avoids $\text{cl}(A)$, then e belongs to a triad that meets $\text{cl}(A)$ in two elements.*

Proof. (i) By Lemma 2.3, the required result follows if we can show that the simplification M'' of M/e is not 3-connected. But e is a coloop of $M|(A \cup e)$, so $(M/e)|A = M|A$, and M'' can be labelled so that its ground set contains A. Thus M'' cannot be 3-connected otherwise (M'', A) contradicts the minimality of (M, A).

(ii) If $M \backslash e$ or M/e is 3-connected, then, since $(M \backslash e)|A = M|A = (M/e)|A$, either $(M \backslash e, A)$ or $(M/e, A)$ contradicts the choice of (M, A). Thus neither $M \backslash e$ nor M/e is 3-connected, so e is essential. But, by (i), the cosimplification of $M \backslash e$ is 3-connected. Thus e must belong to a triad of M.

(iii) When $|A| = 1$, we have that $E(M) = A$; and when $|A| = 2$, we have that $E(M)$ is a circuit of M having at most three elements. In both cases, A is spanning. Thus we may suppose that $|A| \geq 3$. If $|A| = 3$ and A is dependent, then A is either a circuit or a set of parallel elements, so A is spanning; a contradiction. Thus if $|A| = 3$, then A is independent. As A is non-spanning, it follows that

$r(M) \geq 4$. Hence, since M is 3-connected, either $r^*(M) \geq 3$ and so $|E(M)| > 6$, or $r^*(M) = 2$. In the latter case, M is uniform and so, if $e \in E(M) - \mathrm{cl}(A)$, then M/e is also uniform of corank two, so M/e is 3-connected; a contradiction to (i). It remains to consider the case when $|A| \geq 4$. In that case, since A is non-spanning, there is a cocircuit contained in $E(M) - \mathrm{cl}(A)$ and this cocircuit has at least three elements, so $|E(M)| > 6$.

(iv) Since T and T^* meet but are distinct, orthogonality implies that $|T^* - T| = 1$. Let a be the unique element of $T^* - T$. Suppose that $a \notin \mathrm{cl}(A)$. Then A is non-spanning and so, by (iii), $|E(M)| > 6$. Now $M\backslash a$ has $T \cap T^*$ as a 2-element cocircuit. Thus $\{T \cap T^*, E(M\backslash a) - (T \cap T^*)\}$ is a 2-separation of $M\backslash a$. But $T \cap T^*$ spans T so $\{T, E(M\backslash a) - T\}$ is a non-minimal 2-separation of $M\backslash a$; a contradiction to (i). We conclude that $a \in \mathrm{cl}(A)$.

(v) Suppose that $T^* = \{e_1, e_2, a_3\}$ and that $e_i \notin \mathrm{cl}(A)$ for each i in $\{1, 2\}$. Assume that (v) fails and suppose first that $a_3 \in \mathrm{cl}(A)$. Then $a_3 \in T^* \cap (\mathrm{cl}(A) - A)$. Hence, by Lemma 2.5, $T \cap A \neq \emptyset$. We may now assume that $a_3 \notin \mathrm{cl}(A)$. Then $T^* \subseteq E(M) - \mathrm{cl}(A)$. By the dual of Tutte's triangle lemma [13] (see also [9, Lemma 8.4.9]), there is a triangle T of M that meets the triad T^* in exactly two elements. Thus, by (iv), the unique element of $T^* - T$ is in $\mathrm{cl}(A)$. This is a contradiction since $T^* \cap \mathrm{cl}(A) = \emptyset$.

(vi) Choose N to be M or M^* so that both T_1 and T_2 are triangles of N. Assume that $|T_1 \cap T_2| = 2$. Then $N|(T_1 \cup T_2)$ is isomorphic to $U_{2,4}$. Now take e in $(T_1 \cup T_2) - \mathrm{cl}_M(A)$. Then it is straightforward to check that $N\backslash e$ is 3-connected. Thus $M\backslash e$ or $M^*\backslash e$ is 3-connected, so either $M\backslash e$ or M/e is 3-connected; a contradiction to (ii).

(vii) Suppose that $T = \{f_1, f_2, b\}$ for some element b and that, for some i, say $i = 1$, there is no triad meeting $\{f_1, f_2\}$ in $\{f_i\}$. By Tutte's triangle lemma [9, Lemma 8.4.9], there is a triad T^* that contains f_1 and exactly one of b and f_2. Hence $b \notin T^*$ and $f_2 \in T^*$. By (vi), T^* must be the only triad of M that contains f_1. By (i), the cosimplification M' of $M\backslash f_1$ is 3-connected. As the only triad that contains f_1 is T^*, it follows that $M' = M\backslash f_1/f_2$. But, as $|E(M)| \geq 7$, we have $|E(M')| \geq 5$, so M' is simple. Hence M' equals the simplification M'' of M/f_2. Therefore M'' is 3-connected, which is contrary to (i).

(viii) Let i and j be distinct elements of $\{1, 2, 3\}$. Taking $f_1 = e_i$ and $f_2 = e_j$ in (vii), we deduce that M has a triad T_i^* such that $T_i^* \cap \{e_i, e_j\} = \{e_j\}$. Thus $T_i^* \cap T = T - e_i$. By (vi), T_i^* is the unique such triad. Moreover, by (v), T_i^* meets A in a single element, say a_i.

To finish the proof of (viii)(a), we need to check that $\{a_1, a_2, a_3\}$ is a triad of M. Observe that there is a cocircuit D of M such that $a_1 \in D \subseteq (T_1^* \cup T_2^*) - e_3 = \{a_1, a_2, e_1, e_2\}$. As M is 3-connected, it follows that $|D| \geq 3$ and hence $e_i \in D$ for some i. By orthogonality, e_1 and e_2 must both belong to D, otherwise $D \cap T = \{e_i\}$. As T_3^* is the unique triad such that $T_3^* \cap T = \{e_1, e_2\}$, it follows that $D = \{a_1, a_2, e_1, e_2\}$. Thus there is a cocircuit D' of M such that $a_1 \in D' \subseteq (D \cup T_3^*) - e_2 = \{a_1, a_2, a_3, e_1\}$. As $|D' \cap T| \neq 1$ and $|D'| \geq 3$, it follows that $e_1 \notin D'$ and that $D' = \{a_1, a_2, a_3\}$, that is, $\{a_1, a_2, a_3\}$ is a triad of M.

Part (b) is an immediate consequence of [11, Theorem 1.11]. Now we shall prove (c). Let $Z = \{a_1, a_2, a_3, e_1, e_2\}$. We start by proving the following.

4.1.1. *Every circuit of $M|(A \cup \{e_1, e_2\})$ that meets Z must contain Z.*

Let C be such a circuit. Since each a_i is a coloop of $M|A$, the circuit C must meet $\{e_1, e_2\}$. Moreover, as neither e_1 nor e_2 is spanned by A, it follows that $|C - A| \geq 2$ and hence $\{e_1, e_2\} \subseteq C$. As $|T_i^* \cap C| \neq 1$ for each i in $\{1, 2\}$, it follows that $a_i \in C$ for each such i. To prove that $Z \subseteq C$, it remains to show that $a_3 \in C$. Suppose that this is not the case. Then there is a circuit C' of M such that $e_3 \in C' \subseteq (C \cup T) - e_2$. As $C' \cap T_3^* \subseteq \{e_1\}$, it follows that $e_1 \notin C'$. Hence $e_3 \in C' \subseteq A \cup e_3$. This is a contradiction because A does not span e_3. Thus $Z \subseteq C$. We conclude that (4.1.1) holds.

By (viii)(b), M/T, which is isomorphic to $M\backslash e_3/e_1, e_2$, is 3-connected. Since (M, A) is a minimal pair, $M|A \neq (M\backslash e_3/e_1, e_2)|A$. Thus M does indeed have a circuit contained in $A \cup \{e_1, e_2\}$ and meeting $\{e_1, e_2\}$. By (4.1.1), it follows that Z is contained in a series class of $M|(A \cup \{e_1, e_2\})$. Hence $\{a_1, a_2, a_3\}$ is contained in a series class of $[M|(A \cup \{e_1, e_2\})]/\{e_1, e_2\}$, which equals $(M/T)|A$. Finally, for each i in $\{1, 2, 3\}$, each member of $Z - a_i$ is a coloop of $[M|(A \cup \{e_1, e_2\})]\backslash a_i$ so $M|(A - a_i) = [M|(A \cup \{e_1, e_2\})]/\{e_1, e_2\}\backslash a_i = (M/\{e_1, e_2\}\backslash e_3)|(A - a_i) = (M/T)|(A - a_i)$.

(ix) By (vii), if $T \in \mathcal{T}$, there are triads of M such that $T_{1T}^*, T, T_{2,T}^*$ is a chain of M having both its spokes in $E(M) - \mathrm{cl}(A)$. By (iv), it follows that the set A_T of rim elements of this chain is contained in $\mathrm{cl}(A)$. Observe that A_T is contained in a series class or is a set of coloops of $M|\mathrm{cl}(A)$. In both cases, $|A_T \cap A| \geq 2$.

Let a_T be the unique element of $T \cap \mathrm{cl}(A)$. Let T' be an element of $\mathcal{T} - \{T\}$. By Lemma 2.8, $a_T \notin T'$ and so $a_T \neq a_{T'}$.

We shall show next that

4.1.2. *If $a_T \in T_{1T'}^* \cup T_{2T'}^*$, then $A_T \neq A_{T'}$, and $T_{jT}^* = T_{iT'}^*$ for some i and j.*

Assume, without loss of generality, that $a_T \in T_{2T'}^*$. Then, by Lemma 2.8, $T_{2T'}^* = T_{iT}^*$ for some i, say $i = 1$. Hence $T_{1T'}^*, T', T_{2T'}^*, T$ is a chain. Moreover, either $T_{1T'}^*, T', T_{2T'}^*, T, T_{2T}^*$ is a chain, or $T_{2T}^* - T \subseteq T_{1T'}^* \cup T' \cup T_{2T'}^*$. In the first case, $A_T \neq A_{T'}$, as required. Now consider the second case, letting $X = T \cup T' \cup A_{T'}$. By (iii), $|E(M)| \geq 7$, so $|E(M) - X| \geq 1$. Thus, as $X - \{a_T, a_{T'}\}$ and $A_{T'}$ span X in M and M^*, respectively, we have

$$1 \leq r(X) + r^*(X) - |X| \leq 4 + 3 - 6 = 1.$$

Since M is 3-connected, it follows that $|E(M)| = 7$ and $r(X) = 4$. But $r(T \cup T') = 3$ and $|E(M) - (T \cup T')| = 2$, so M has a cocircuit of size at most two; a contradiction. We conclude that (4.1.2) holds. It follows immediately from this that (ix)(a) holds.

If $A_T \cap A_{T'} = \emptyset$, then

$$|(A_T \cup A_{T'}) \cap A| = |A_T \cap A| + |A_{T'} \cap A| \geq 4.$$

Hence (ix)(b) holds in this case, so we may suppose that $A_T \cap A_{T'} \neq \emptyset$. Let $S = A_T \cup A_{T'}$. Then, in $M|\mathrm{cl}(A)$, the set S is contained in a series class or is a set of coloops. Thus, by orthogonality and (a), $|A \cap S| \geq |S| - 1 \geq 3$. To complete the proof of (b), we now suppose that $|S \cap A| = 3$. Then $|S| = 4$, and S must be contained in a series class of $M|\mathrm{cl}(A)$ otherwise the unique element of $S - A$ is a coloop of $M|\mathrm{cl}(A)$ contained in $\mathrm{cl}(A) - A$; a contradiction. Thus $|A_T \cap A_{T'}| = 2$. But, by applying (4.1.2) to both a_T and $a_{T'}$, we have that either $T_{jT}^* = T_{iT'}^*$ for some i and j, or $a_T \notin A_{T'}$ and $a_{T'} \notin A_T$. In the latter case, $A_T - \{a_T\} = A_{T'} - \{a_{T'}\}$, that is, $A_T - T = A_{T'} - T'$ and we have established that (ix)(b) holds.

(x) Let $\mathcal{T}'$ be the set of triangles T of M such that $T - \mathrm{cl}(A) \neq \emptyset$. As A does not span M, there is a cocircuit of M that avoids $\mathrm{cl}(A)$. For each such cocircuit

D, by Theorem 2.7, there are distinct triangles T_1' and T_2' such that $T_i' \cap D \neq \emptyset$. Hence $|\mathcal{T}'| \geq 2$.

Suppose that $\mathcal{T}'$ has a member T such that $T \cap \mathrm{cl}(A) = \emptyset$. Let $T = \{e_1, e_2, e_3\}$. By (viii), for each i in $\{1, 2, 3\}$, there is a triad T_i^* of M and an element a_i of A such that $T_i^* = (T - e_i) \cup a_i$. Thus a_i is a coloop of $M|A$ and hence, $|A| = 3$ because $r(A) \leq 3$ by hypothesis. Therefore $A = \{a_1, a_2, a_3\}$. If $f \in \mathrm{cl}(A) - A$, then there is a circuit C of M such that $f \in C \subseteq A \cup f$. Thus $a_i \in C$ for some i, so $C \cap T_i^* = \{a_i\}$, which is contrary to orthogonality. Hence $\mathrm{cl}(A) = A$. Let T' be in $\mathcal{T}' - \{T\}$. Next we shall prove that $T' \cap A \neq \emptyset$. Assume the contrary. Then T' avoids T otherwise $|T' \cap T_i^*| = 1$ for some i. Now, applying (viii)(c) to the triangle T', it follows that A is contained in a series class of $(M/T')|A$ and hence, as $|A| = 3$, we deduce that A is a circuit of M/T'. But T_1^* is a triad of M/T', a contradiction since $|A \cap T_1^*| = 1$. We conclude that T' does indeed meet A. Thus we may assume that $a_1 \in T'$. By orthogonality, e_2 or e_3, say e_2, belongs to T'. By (vi), $T \cap T' = \{e_2\}$. As $T' \cap T_3^* \neq \{e_2\}$, it follows that $a_3 \in T'$ and $T' = \{a_1, e_2, a_3\}$. This is a contradiction because it implies that $e_2 \in \mathrm{cl}(A)$, yet $\mathrm{cl}(A) = A$.

We may now assume that every T in $\mathcal{T}'$ meets $\mathrm{cl}(A)$. Thus, in the notation of (ix), $\mathcal{T}' = \mathcal{T}$. Let $T \in \mathcal{T}$. Then $A_T \cup T$ is the union of two triads of M. Thus, in $M|\mathrm{cl}(A)$, either (a) A_T is contained in a series class, or (b) A_T is a set of coloops. But, in the latter case, A_T is an independent set of size three. Since $A_T \subseteq \mathrm{cl}(A)$, we deduce that A_T spans $\mathrm{cl}(A)$. Thus $A_T = \mathrm{cl}(A) = A$. But $|\mathcal{T}'| \geq 2$ so there is a T' in $\mathcal{T} - T$. As $A_{T'}$ is a 3-element subset of the 3-element set $\mathrm{cl}(A)$, we have $A_{T'} = A = A_T$; a contradiction to (ix)(a). Hence we may assume that (a) holds. Then, as $|A_T| = 3$, either A_T is a basis of A, or A_T is a triangle. In the second case, letting $X = A_T \cup T$, we have $r(X) + r^*(X) - |X| \leq 1$. This contradicts the fact that M is 3-connected since $|X| = 5$ and, by (iii), $|E(M)| \geq 7$. We conclude that A_T is a basis of A and it is not difficult to see that $\mathrm{cl}(A)$ is a 4-circuit of M.

Now let T and T' be distinct triangles in $\mathcal{T}$ and let $Z = \mathrm{cl}(A) \cup T \cup T'$. Since each of $\mathrm{cl}(A)$, T, and T' is a circuit and $\mathrm{cl}(A) - (T \cup T')$ is non-empty, we deduce that $r(Z) \leq |Z| - 3$. Moreover, each element of $T \cup T'$ is in a triad with two elements of $\mathrm{cl}(A)$. Thus $r^*(Z) \leq |\mathrm{cl}(A)| = 4$. Hence

$$r(Z) + r^*(Z) - |Z| \leq (|Z| - 3) + 4 - |Z| = 1.$$

Since $\{Z, E(M) - Z\}$ is not a 2-separation of M, it follows that $|E(M) - Z| \leq 1$ and so $|E(M)| \leq 9$. But A_T and $A_{T'}$ are distinct 3-element subsets of the 4-element set $\mathrm{cl}(A)$, so $\mathrm{cl}(A) = A_T \cup A_{T'}$. Hence every element of $\mathrm{cl}(A)$ is in a triad of M. But, by (ii), every element of $E(M) - \mathrm{cl}(A)$ is also in a triad of M. Hence M is minimally 3-connected. Moreover, $r(M) \geq 4$; and $|E(M) - \mathrm{cl}(A)| \geq 4$ otherwise $E(M) - \mathrm{cl}(A)$ is a triad avoiding A contradicting (v). Thus $|E(M)| \geq 8$.

Suppose that $r(M) = 4$. Then, by Oxley [8], since M is minimally 3-connected, $|E(M)| \leq 8$ with equality only if M is a wheel or a whirl. Therefore $|E(M)| = 8$. Since the only 4-circuit in a rank-four wheel or whirl that is a flat of the matroid is the rim of the rank-four wheel, the result holds when $r(M) = 4$.

We may now assume that $r(M) \geq 5$. Since every cocircuit avoiding the 4-circuit $\mathrm{cl}(A)$ has size at least four and $|E(M)| \leq 9$, it follows that $r(M) = 5$ and $|E(M)| = 9$. Thus $E(M) - \mathrm{cl}(A)$ contains at least two cocircuits of M. By elimination, it follows that every 4-element subset of $E(M) - \mathrm{cl}(A)$ is a cocircuit. But T meets $E(M) - \mathrm{cl}(A)$ in exactly two elements, so $E(M) - \mathrm{cl}(A)$ contains a cocircuit meeting T in a single element; a contradiction.

(xi) From (ii), there is a triad T^* of M such that $e \in T^*$. Moreover, we may assume that $|T^* \cap \mathrm{cl}(A)| \leq 1$ otherwise the result holds. Thus $|T^* - \mathrm{cl}(A)| \geq 2$. Hence, by Tutte's triangle lemma, M has a triangle T that meets T^* in exactly two elements, one of which is e. Let $T = \{e, f, g\}$ where $f \notin \mathrm{cl}(A)$. We may assume that $g \in \mathrm{cl}(A)$. By (vii), M has a triad T^*_e that meets $\{e, f\}$ in $\{e\}$. Orthogonality implies that $g \in T^*_e$, and (iv) implies that $T^*_e - T \subseteq \mathrm{cl}(A)$. Thus $|T^*_e \cap \mathrm{cl}(A)| = 2$, as desired. □

4.2. Lemma. *Let (M, A) be a minimal pair and T be a triangle of M that avoids $\mathrm{cl}(A)$. If a is a member of A that belongs to a triad meeting T, then $(M/T, A - a)$ is a minimal pair.*

Proof. Let $T = \{e_1, e_2, e_3\}$. Then, by (viii) of Lemma 4.1, for each i, there is a unique triad T^*_i of M such that $T^*_i = (T - e_i) \cup a_i$ for some a_i in A. Thus $a = a_i$ for some i and, by symmetry, we may assume that $i = 3$. Let $M' = M/T$ and $A' = A - a_3$. By Lemma 4.1(viii)(b), M' is 3-connected. We may suppose that (M', A') is not a minimal pair. Then there is a 3-connected minor N' of M' such that $N'|A' = M'|A'$ and (N', A') is a minimal pair. Let $N' = M' \backslash X/Y$. Observe that $a_3 \notin Y$ because Lemma 4.1(viii)(b) implies that a_3 is spanned by A' in M'.

Suppose that $a_3 \in X \cup E(N')$. Then, since $\{a_1, a_2, a_3\}$ is a triad of M', either

(a) $a_3 \in E(N')$ and $\{a_1, a_2, a_3\}$ is a triad of N'; or
(b) $\{a_1, a_2, a_3\}$ properly contains a cocircuit of N'.

In case (b), since N' is 3-connected and $\{a_1, a_2\}$ has rank two in it, we deduce that N' must be a triangle. As a_1 and a_2 are coloops of $M'|A'$ and $M'|A' = N'|A'$, it follows that $A' = \{a_1, a_2\}$ and $|A| = 3$. This cannot happen by (x) of Lemma 4.1 since the rank-four wheel has no triangle avoiding the closure of its rim. Hence (a) must hold. Now, by Lemma 4.1(viii)(b), $M^* = P_\Delta(M(K_4), M^* \backslash T)$ where $\Delta = \{a_1, a_2, a_3\}$ and $M(K_4)$ is labelled as in Figure 1; and $(N')^* = (M^* \backslash T)/X \backslash Y$. Since Δ is a triangle of $(N')^*$, it follows that

$$M^*/X \backslash Y = P_\Delta(M(K_4), M^* \backslash T/X \backslash Y) = P_\Delta(M(K_4), (N')^*).$$

Let $N^* = M^*/X \backslash Y$. Since $(N')^*$ and $M(K_4)$ are 3-connected having at least three common elements, by [10], N^* is 3-connected. Hence N is 3-connected.

We shall show next that

4.2.1. $N|A' = M|A'$.

To see this, first note that $M'|A' = M|A'$ by Lemma 4.1(viii)(c). Moreover, by assumption, $N'|A' = M'|A'$. Thus

$$r_M(A') = r_{M'}(A') = r_{N'}(A').$$

But, since N' is a minor of N, and N is a minor of M, we have

$$r_M(A') \geq r_N(A') \geq r_{N'}(A').$$

Therefore equality holds throughout the last line, so $r_N(A') = r_M(A')$. As N is a minor of M, it is now easy to check that $N|A' = M|A'$. Hence (4.2.1) holds.

Since a_3 is a coloop of both $N|A$ and $M|A$, it follows from (4.2.1) that $N|A = M|A$. Therefore, as (M, A) is a minimal pair, $N = M$ and so $X \cup Y$ is empty and $N' = M'$. Thus (M', A') is a minimal pair. □

4.3. Lemma. *Suppose that (M, A) is a minimal pair in which A is non-spanning such that (a) $M \setminus \mathrm{cl}(A)$ has no triangles; and (b) every triad with at least two elements in $\mathrm{cl}(A)$ contains at least two elements of A. Let a be an element of A that is in no triangles of M, and N be a minimal 3-connected minor of M/a for which $N|(A-a) = (M/a)|(A-a)$. Then*

(i) *$N = M/a$; or*
(ii) *$N = M/a\backslash e$ for some element e of $E(M) - A$ such that $\{a, e\}$ is contained in a triad of M whose third element is in A; or*
(iii) *$N = M/\{a, e\}\backslash e'$ for some elements e and e' of $E(M) - \mathrm{cl}(A)$ such that $\{e, a_1, e'\}$ is a triangle of M and $\{a, e, a_1\}, \{e, a_1, e'\}, \{a_1, e', a_2\}$ is a fan of M for some elements a_1 and a_2 of A.*

Proof. Suppose that $N = (M/a)\backslash X/Y$. We show first that

4.3.1. $Y \cap \mathrm{cl}(A) = \emptyset$.

Assume that (4.3.1) fails and let y be an element of $Y \cap \mathrm{cl}(A)$. As $N|(A-a) = (M/a)|(A-a)$ and N is a minor of $M/\{a, y\}$, it follows that

$$r_{M/a}(A-a) = r_N(A-a) \le r_{M/\{a,y\}}(A-a) = r_{M/a}((A-a) \cup y) - r_{M/a}(y).$$

Thus $y \notin \mathrm{cl}_{M/a}(A-a)$ so $y \notin \mathrm{cl}_M(A)$; a contradiction. Hence (4.3.1) holds.

Now choose X and Y such that $|X|$ is as large as possible. Note that

4.3.2. *$Y \cup a$ is independent in M.*

If not, there is a circuit C of M such that $C \subseteq Y \cup a$. Take $y \in C - a$. Then y is a loop of $M/[(Y-y) \cup a]$. Hence $N = (M/a)\backslash(X \cup y)/(Y-y)$, which is contrary to the choice of X and Y.

We show next that we may also suppose that

4.3.3. $|E(N)| \ge 4$.

Suppose not. Then $|E(N)| \le 3$ and so, as N is 3-connected, $r(N) \le 2$. Thus

$$r_M(A) = r_{M/a}(A-a) + 1 = r_N(A-a) + 1 \le r(N) + 1 \le 3$$

where the second equality holds since $N|(A-\{a\}) = (M/a)|(A-\{a\})$. But now, by (x) of Lemma 4.1, we conclude that M is isomorphic to a rank-four wheel having $\mathrm{cl}(A)$ as its rim. Hence a is in a triangle of M; a contradiction.

Next we prove the following.

4.3.4. *Let $H = M\backslash X_1/Y_1$ where $X_1 \subseteq X$ and $Y_1 \subseteq Y \cup a$. Then H cannot have a cocircuit D such that $|D| \le 2$ and $D \subseteq \mathrm{cl}(A) - a$.*

Suppose that such a cocircuit D does exist. We shall prove that $D \cap A \neq \emptyset$. Assume that $D \cap A = \emptyset$ and let d be an element of D. As $d \notin A$, there is a circuit C of M such that $d \in C \subseteq A \cup d$. Observe that, since $A - a \subseteq E(H)$ and $a \notin X_1$, the set C contains a circuit C' of H such that $d \in C' \subseteq A \cup d$. Hence $C' \cap D = \{d\}$, which is contrary to orthogonality. Thus $D \cap A \neq \emptyset$. Let a' be an element of $D \cap A$. Observe that a' must belong to a cocircuit D' of N such that $D' \subseteq D$. Thus $|D'| \le |D| \le 2$, which contradicts (4.3.3) since N is 3-connected. Hence (4.3.4) holds.

The second result that we need about a minor H of M having N as a minor is the following.

4.3.5. *If H is not 3-connected, then there is a k-separation $\{Z, W\}$ of H with k in $\{1, 2\}$ such that $|Z \cap E(N)| \leq 1$ and Z is closed in both H and H^*.*

As H is not 3-connected, it has a k-separation $\{Z, W\}$ for some k in $\{1, 2\}$. But N is a 3-connected matroid and is a minor of H, so

$$\min\{|Z \cap E(N)|, |W \cap E(N)|\} \leq 1.$$

In particular, we may assume that

$$|Z \cap E(N)| \leq 1. \tag{1}$$

Now choose such a k-separation satisfying (1) so that $|Z|$ is as large as possible. Note that, since $|E(N)| \geq 4$, we must have that

$$|W \cap E(N)| \geq 3. \tag{2}$$

If Z is closed in both H and H^*, then (4.3.5) follows. Thus, we may assume that there is an element w of W such that w is spanned by Z in H or H^*. Hence $r_H(Z) = r_H(Z \cup w)$ or $r_{H^*}(Z) = r_{H^*}(Z \cup w)$, and so

$$r_H(Z) + r_{H^*}(Z) - |Z| \geq r_H(Z \cup w) + r_{H^*}(Z \cup w) - |Z \cup w|.$$

Thus $\{Z \cup w, W - w\}$ is a j-separation of H for some j in $\{1, 2\}$. Again, since N is 3-connected, we have $\min\{|(Z \cup w) \cap E(N)|, |(W - w) \cap E(N)|\} \leq 1$. It follows by (2) that $|(Z \cup w) \cap E(N)| \leq 1$. Therefore $\{Z \cup w, W - w\}$ contradicts the choice of $\{Z, W\}$ and we conclude that (4.3.5) holds.

4.3.6. *If T^* is a triad of M such that $e \in T^* - \mathrm{cl}(A)$ and $T^* - e \subseteq \mathrm{cl}(A)$, then $T^* - e \subseteq A$.*

To see this, note that $T^* \cap \mathrm{cl}(A) = T^* - e$. Hence $|T^* \cap \mathrm{cl}(A)| = 2$ and (4.3.6) follows by hypothesis (b).

4.3.7. *If T is a triangle of M such that $T - \mathrm{cl}(A)$ is non-empty, then $T \cap \mathrm{cl}(A)$ contains exactly one element, a_T, and $a_T \in A$.*

As $T - \mathrm{cl}(A)$ is non-empty but $M \backslash \mathrm{cl}(A)$ has no triangles, it follows that $1 \leq |T - \mathrm{cl}(A)| \leq 2$. But $\mathrm{cl}(A)$ does not span T, so $|T - \mathrm{cl}(A)| = 2$. Let e be an element of $T - \mathrm{cl}(A)$. By (xi) of Lemma 4.1, there is a triad T^* of M such that $e \in T^*$ and $T^* - e \subseteq \mathrm{cl}(A)$. By (4.3.6), $T^* - e \subseteq A$. By orthogonality, $T^* \cap T \neq \{e\}$. Thus the unique element a_T of $T \cap (T^* - e)$ must belong to A, and (4.3.7) holds.

Recall that $N = (M/a) \backslash X/Y$. To establish Lemma 4.3, we shall show that both X and Y must be small. We begin by considering X.

4.4. Lemma. (i) *If $x \in X$, then there is a triad T^*_x of M such that $T^*_x = \{x, a, b\}$ for some $b \in A$.*

(ii) $|X| \leq 1$.

Proof. The proof of Lemma 4.4(i) is long and will be divided into a number of steps. We shall argue by contradiction. Thus suppose that (i) fails, that is, T^*_x does not exist for some x in X. Let $H = M \backslash x$ and observe that H is not 3-connected since (M, A) is a minimal pair. Therefore H has a 2-separation $\{Z, W\}$ satisfying (4.3.5). Hence $Z \cap E(N)$ has at most one element. Moreover, when this element exists, we shall denote it by n. It is not difficult to see that

4.4.1. *$Z \cap A \subseteq \{a, n\}$, where n may not exist.*

Next we show that

4.4.2. *$A \cup W$ is a spanning set of M.*

Suppose that $W \cup A$ does not span M. Then there is a cocircuit D of M that avoids $\mathrm{cl}(A \cup W)$. Since D must also avoid $\mathrm{cl}(A)$, Lemma 4.1(ii) implies that every element of D is essential. Thus, by Theorem 2.7, M has two distinct triangles T_1 and T_2 meeting D. By (4.3.7), each T_i contains a unique element a_{T_i} of $\mathrm{cl}(A)$ and $a_{T_i} \in A$. Moreover, by Lemma 4.1(ix), $a_{T_1} \neq a_{T_2}$. But, for each i, we have $T_i - a_{T_i} \subseteq D$. Moreover, $D \subseteq Z$ since D avoids $\mathrm{cl}(W)$. Since, by (4.3.5), Z is closed in $M \backslash x$, it follows that $a_{T_i} \in Z$ for each i. Hence, by (4.4.1), $\{a_{T_1}, a_{T_2}\} \subseteq Z \cap A \subseteq \{a, n\}$. Therefore $a = a_{T_j}$ for some j, so a is in a triangle; a contradiction to the hypothesis.

4.4.3. $2 \leq r(Z) \leq 3$. *Moreover, if $r(Z) = 3$, then n exists and is in A, and both a and n belong to Z and are coloops of $M|(W \cup \{a,n\})$; and if $a \in W$ and $r(Z) = 2$, then n exists and is in A and n is a coloop of $M|(W \cup n)$.*

By (4.4.1), $W \cup A \subseteq W \cup \{a, n\}$ where n may not exist. As $\{Z, W\}$ is a 2-separation of $M \backslash x$, we have $\min\{|Z|, |W|\} \geq 2$ and $r(Z) + r(W) = r(M) + 1$. But $M \backslash x$ is simple, so

$$2 \leq r(Z) = r(M) - r(W) + 1. \tag{3}$$

Moreover, by (4.4.2),

$$r(M) = r(W \cup A) = r(W \cup (A - W)) \leq r(W) + |A - W|.$$

Hence

$$r(W \cup A) - r(W) = r(M) - r(W) \leq |A - W|. \tag{4}$$

Substituting for $r(M) - r(W)$ into (3), we get

$$2 \leq r(Z) \leq |A - W| + 1. \tag{5}$$

Since $A - W \subseteq \{a, n\}$, where n may not exist, we deduce that $2 \leq r(Z) \leq 3$. Moreover, if $r(Z) = 3$, then $|A - W| = 2$ and equality holds in (4). Hence n exists, both a and n are in Z, and both are coloops of $M|(W \cup \{a, n\})$. On the other hand, if $a \in W$, then $|A - W| = 1$, so n exists and equality holds throughout both (5) and (4). Thus $r(Z) = 2$, and n is a coloop of $M|(W \cup n)$.

4.4.4. *If $|Z| \geq r(Z) + 1$, then $(Z - A) \cap \mathrm{cl}(A)$ is empty, and n exists and is in A.*

Suppose first that $r(Z) = 2$. Then $M|Z$ is isomorphic to $U_{2,|Z|}$. As a does not belong to a triangle, it follows that $a \in W$. Hence, by (4.4.3), n exists and is in A. Thus, by (4.4.1), $Z - A = Z - n$. Now suppose that A spans an element z of $Z - n$. Then $\{n, z\} \subseteq \mathrm{cl}(A)$ and A spans Z because $\{n, z\}$ is a basis for Z. As $\mathrm{cl}(A) \cap Y = \emptyset$ by (4.3.1), $Z \cap Y = \emptyset$. But $Z - n$ avoids $E(N)$. Hence $Z - n \subseteq X$. Thus N is a minor of $M \backslash (Z - n)$ which equals $M|(W \cup n)$. But n is a coloop of the last matroid by (4.4.3), so n is a coloop of the 3-connected matroid N; a contradiction. We conclude that A cannot span any element of $Z - A$, and so $(Z - A) \cap \mathrm{cl}(A) = \emptyset$ when $r(Z) = 2$.

We may now assume that $r(Z) \geq 3$. Then, by (4.4.3), $r(Z) = 3$, n exists, and a and n both belong to $Z \cap A$. Thus, by (4.4.1), $Z - A = Z - \{a, n\}$. Suppose that A spans an element z of $Z - \{a, n\}$. Then $\{a, n, z\} \subseteq \mathrm{cl}(A)$ and A spans Z because $\{a, n, z\}$ is a basis for Z, since, by hypothesis, a does not belong to a triangle of M. As $\mathrm{cl}(A) \cap Y = \emptyset$ by (4.3.1), $Z \cap Y = \emptyset$. But $Z - \{a, n\}$ avoids $E(N)$ so

$Z-\{a,n\} \subseteq X$. Thus N is a minor of $M\backslash(Z-\{a,n\})$, which equals $M|(W\cup\{a,n\})$. Since n is a coloop in the last matroid by (4.4.3), n is a coloop of N; a contradiction. Hence A cannot span any element of $Z-A$ and so $(Z-A)\cap \mathrm{cl}(A)=\emptyset$.

4.4.5. $r(Z)=|Z|$.

Suppose that $|Z|\geq r(Z)+1$. Then $M|Z$ has a circuit, C. Observe that $C-A=C-\{a,n\}$ since, by (4.4.4), n exists and belongs to A and by (4.4.1), $Z\cap A\subseteq\{a,n\}$. Now $|C-\{a,n\}|\geq 2$, otherwise $\{a,n\}\subseteq C$ and $|C|=3$ which is contrary to the fact that a does not belong to a triangle. Let z_1 and z_2 be distinct elements of $C-A$, and suppose that $z\in\{z_1,z_2\}$. Then $z\in Z-A$, so, by (4.4.4), $z\notin \mathrm{cl}(A)$. Moreover, by (xi) of Lemma 4.1, M has a triad T^* that contains z and meets $\mathrm{cl}(A)$ in two elements. By (4.3.6), T^* must meet A in two elements. By orthogonality, $|T^*\cap C|\geq 2$. Hence $|T^*-C|\leq 1$ and so $T^*\subseteq Z$ by (4.3.5). Thus $T^*\cap A\subseteq Z\cap A\subseteq\{a,n\}$ by (4.4.1). Hence $T^*=\{z,a,n\}$. In particular, the triads $\{z_1,a,n\}$ and $\{z_2,a,n\}$ contradict (vi) of Lemma 4.1.

Next we complete the proof of the first part of Lemma 4.4 by showing that T_x^* exists, thereby obtaining a contradiction since we have assumed that T_x^* does not exist.

4.4.6. *T_x^* exists.*

Suppose first that $x\in X-\mathrm{cl}(A)$. Then, by (xi) of Lemma 4.1, there is a triad T^* of M such that $T^*-x\subseteq \mathrm{cl}(A)$ and hence, by (4.3.6), $T^*-x\subseteq A$. If $a\in T^*$, we can take $T^*=T_x^*$ and (4.4.6) holds. Thus we may assume that $a\notin T^*$. Then T^*-x is a 2-cocircuit of $M\backslash x/a$. As N is a minor of $M\backslash x/a$ and $T^*-x\subseteq \mathrm{cl}(A)-a$, the cocircuit T^*-x contradicts (4.3.4). We conclude that, for every element x of $X-\mathrm{cl}(A)$, the triad T_x^* exists.

Now suppose that $x\in \mathrm{cl}(A)-A$. By (4.4.5), $r(Z)=|Z|$. As $\{Z,W\}$ is a 2-separation for $M\backslash x$, we have, by Lemma 2.4, that if z and z' are distinct elements of Z, then $\{x,z,z'\}$ is a cocircuit of M. But $x\in\{x,z,z'\}\cap(\mathrm{cl}(A)-A)$, so, by Lemma 2.5, $|\{x,z,z'\}\cap \mathrm{cl}(A)|\geq 2$. Hence, by hypothesis (b), $|\{x,z,z'\}\cap A|\geq 2$. As $x\notin A$, it follows that both z and z' belong to A. Since these two elements were arbitrarily chosen in Z, it follows that $Z\subseteq A$. Thus $Z\cap A=Z$. By (4.4.1), $Z\cap A\subseteq\{a,n\}$ and hence $Z=\{a,n\}$ since $|Z|\geq 2$. In this case, we can take $T_x^*=\{x,a,n\}$ thereby completing the proof of (4.4.6) and hence that of Lemma 4.4(i).

To prove the second part of the lemma, we shall argue by contradiction. Suppose that x and x' are different elements of X. By (i), the triads T_x^* and $T_{x'}^*$ both exist. Let $T_x^*=\{a,x,b\}$ and $T_{x'}^*=\{a,x',b'\}$ be as in Lemma 4.4. Thus $\{b,b'\}\subseteq A$.

We prove next that $b\neq b'$. Assume the contrary. Then, by (vi) of Lemma 4.1, both x and x' belong to $\mathrm{cl}(A)$. Moreover, $M^*|\{a,x,x',b\}$ is isomorphic to $U_{2,4}$. Hence $\{x,x',b\}$ is a triad of M containing two elements of $\mathrm{cl}(A)$ but just one element of A; a contradiction to hypothesis (b). Thus $b\neq b'$.

Observe that $\{b,b',a\}$ is contained in a series class or is a set of coloops of $M\backslash\{x,x'\}$. Thus $\{b,b'\}$ contains a cocircuit of $(M/a)\backslash\{x,x'\}$ which is contrary to (4.3.4). □

Recall that $N=(M/a)\backslash X/Y$. We have just shown that $|X|\leq 1$. Next we consider $|Y|$.

4.5. Lemma. *If $y\in Y$, then y belongs to a triangle of M.*

Proof. Suppose that y is in no triangles of M. By (4.3.1), $y \notin \mathrm{cl}(A)$. Thus, if $H = M/y$, then H has a 2-separation $\{Z, W\}$. By (4.3.5), we may assume that $\{Z, W\}$ is chosen such that $|Z \cap E(N)| \leq 1$ and Z is closed in both H and H^*. As before, if $Z \cap E(N)$ is non-empty, its unique element will be denoted by n. Thus

$$Z \cap A \subseteq \{a, n\}. \tag{6}$$

As M is 3-connected and $\{Z, W\}$ is a 2-separation for M/y, it follows that

$$y \in \mathrm{cl}(Z) \cap \mathrm{cl}(W) \tag{7}$$

We shall prove next that

4.5.1. *$Z - X$ is independent in $M/y\backslash X$.*

We argue by contradiction. Thus assume that M/y has a circuit C that is contained in $Z - X$. By (4.3.2), it follows that $C - (Y \cup a) \neq \emptyset$. Hence

$$\emptyset \neq C - (Y \cup a) \subseteq Z - (X \cup Y \cup a) \subseteq Z \cap E(N) \subseteq \{n\}.$$

Thus n exists and is in C, and $C-(Y\cup a) = \{n\}$. Hence n is a loop of $(M/y)/(C-n)$ and $y \cup (C - n) \subseteq Y \cup a$ so N is a minor of $(M/y)/(C - n)$. Therefore n is a loop of N; a contradiction. We conclude that (4.5.1) holds.

Now, we shall prove that

4.5.2. *$|Z| \geq 3$ and X contains a unique element x. Moreover, $x \in Z$ and, for every 2-element subset Z_0 of $Z - X$, the set $Z_0 \cup x$ is a triad of M.*

The fact that $|Z| \geq 3$ follows because Z spans y in M by (7), and y does not belong to a triangle. Now, by Lemma 4.4, $|X| \leq 1$. Moreover, since $|Z \cap E(N)| \leq 1$ and $|E(N)| \geq 4$, we have

$$|W| \geq |E(N) - Z| \geq |E(N)| - 1 \geq 3.$$

Thus $|W - X| \geq 2$ and, since $|Z| \geq 3$, we also have that $|Z - X| \geq 2$. Hence $\{Z - X, W - X\}$ is a 2-separation for $M/y\backslash X$. Therefore, by (4.5.1) and Lemma 2.4, if Z_0 is a 2-element subset of $Z - X$, then Z_0 contains a cocircuit of $M/y\backslash X$. Thus $Z_0 \cup X$ contains a cocircuit of M/y and hence of M. But every cocircuit of M has at least three elements, so $Z_0 \cup X$ is a cocircuit of M and $|X| = 1$ since, by Lemma 4.4, $|X| \leq 1$. Moreover, $X \subseteq Z$ since Z is closed in H^*.

Now, we shall show that

4.5.3. $|Z| = 3$.

Suppose that $|Z| \geq 4$. By (4.5.2), there is a unique element x in X, and $M^*|Z$ is isomorphic to $U_{2,|Z|}$. If $x \notin \mathrm{cl}(A)$, it follows that M has two triads that contradict (vi) of Lemma 4.1. Thus $x \in \mathrm{cl}(A)$. Now let Z_0 be a 2-element subset of $Z - x$. By (4.5.2), $x \cup Z_0$ is a triad of M. Since $x \in \mathrm{cl}(A) - A$, it follows by Lemma 2.5 that $|(x \cup Z_0) \cap \mathrm{cl}(A)| \geq 2$. Thus, by hypothesis (b), $|(x \cup Z_0) \cap A| \geq 2$. As $x \notin A$, it follows that $Z_0 \subseteq A$. As Z_0 was chosen arbitrarily, we deduce that $Z - x \subseteq A$. Thus $Z - x = Z \cap A$. But, by (6), $Z \cap A \subseteq \{a, n\}$. Thus $3 \leq |Z - x| = |Z \cap A| \leq 2$; a contradiction. Hence (4.5.3) holds.

Next we prove the following.

4.5.4. *$Z \cup y$ is a circuit of M, the element n exists and is in A, and Z equals $\{x, a, n\}$ and is a triad of M.*

As Z spans y by (7), it follows that $Z \cup y$ is a circuit of M, since y does not belong to a triangle and $|Z| = 3$. By Lemma 4.4, there is a triad T_x^* of M such that $x \in T_x^*$ and $T_x^* - x \subseteq A$. By orthogonality, since $x \in Z$, it follows that $|T_x^* \cap (Z \cup y)| \geq 2$. As $y \notin A$, it follows that T_x^* is a triad of H, which equals M/y. Since Z is closed in H^*, we deduce that $T_x^* \subseteq Z$ because $1 \geq |T_x^* - (Z \cup y)| = |T_x^* - Z|$. Hence $T_x^* = Z$. Thus $T_x^* - x = T_x^* \cap A = Z \cap A \subseteq \{a, n\}$, by (6). Hence n exists and is in A, and $T_x^* = \{a, n, x\}$. We conclude that (4.5.4) holds.

Now $y \notin \mathrm{cl}(A)$ by (4.3.1). Thus, by (xi) of Lemma 4.1, there is a triad T^* of M such that $y \in T^*$ and $T^* - y \subseteq \mathrm{cl}(A)$. By (4.5.4), Z is also a triad of M and $Z \cup y$ is a circuit of M. By (vi) of Lemma 4.1, $|Z \cap T^*| = 1$. Moreover, M/x is not 3-connected since $x \notin \mathrm{cl}(A)$ otherwise $(Z \cup y) - \mathrm{cl}(A) = \{y\}$, which is contrary to (4.3.1). Thus we may apply the dual of Lemma 2.6 to M to deduce that x is in some triangle T of M. By orthogonality, $|T \cap Z| \geq 2$ and hence $|T - Z| \leq 1$. As Z is closed in M/y, it follows that $T \subseteq Z$. Hence $T = Z$ and Z is both a triangle and a triad of the 3-connected matroid M. This contradiction implies that Lemma 4.5 holds. □

4.6. Lemma. *$|Y| \leq 1$ and if $|Y| = 1$, then $|X| = 1$. Moreover, if $Y = \{y\}$ and $X = \{x\}$, then there is an element a' of $A - a$ such that $\{x, a', a\}$ is a triad of M and $\{x, y, a'\}$ is a triangle of M.*

Proof. Suppose that $y \in Y$. By Lemma 4.5, M has a triangle T_y containing y. Now, by (4.3.1), $y \notin \mathrm{cl}(A)$. Thus, by (4.3.7), T_y contains a unique element a' of A. Moreover, $a' \neq a$ since, by hypothesis, a is in no triangles. Hence $a' \in E(N)$. Let $T_y = \{y, a', d\}$ where d is some element of $E(M)$. Then a' and d are parallel in M/y. But N is a 3–connected minor of M/y and has no circuits of size less than three. Hence $d \notin E(N)$. Now $d \notin Y$, otherwise a' is a loop of N. Therefore $d \in X$. Thus, by Lemma 4.4, d is the unique element x of X, and M has a triad T_x^* where $T_x^* = \{x, a, b\}$ for some b in A. Since $x \in T_x^* \cap T_y$, orthogonality implies that $a' = b$ as a is in no triangles. Thus $T_y = \{y, x, b\}$. But y was arbitrarily chosen in Y. Hence if $y' \in Y - y$, then $M|\{y, y', x, b\} \cong U_{2,4}$ and so $\{y, y', x\}$ is a triangle of M meeting the triad $\{x, a, b\}$ in a single element, x; a contradiction. We conclude that $Y - y$ is empty. Hence $|Y| \leq 1$. Moreover, if $|Y| = 1$, say $Y = \{y\}$, then $X = \{x\}$ and the required triangle and triad of M exist. □

We are now ready to finish the proof of Lemma 4.3. We have $N = (M/a)\backslash X/Y$. By Lemma 4.4, $|X| \leq 1$, and, by Lemma 4.6, $|Y| \leq 1$ with equality in the latter implying equality in the former. Hence we have three possibilities:

(i) $|Y| = 0 = |X|$;
(ii) $|Y| = 0$ and $|X| = 1$; and
(iii) $|Y| = 1 = |X|$.

The first possibility means that (i) of Lemma 4.3 holds. Moreover, by Lemma 4.4, the second possibility implies that (ii) of Lemma 4.3 holds. Thus it remains only to consider when the third possibility holds. Hence let $Y = \{y\}$ and $X = \{x\}$. By Lemma 4.6, for some a' in $A - a$, the matroid M has a triad T_1^* and a triangle T such that $T_1^* = \{a, x, a'\}$ and $T = \{x, y, a'\}$. Since $y \notin \mathrm{cl}(A)$, the triangle T implies that $x \notin \mathrm{cl}(A)$. Moreover, by (xi) of Lemma 4.1 and (4.3.6), there is a triad T_2^* of M containing y such that $T_2^* - y \subseteq A$. Thus, by orthogonality with the triangle $\{x, y, a'\}$, it follows that $T_2^* = \{a', y, a''\}$ for some a'' in A. Hence either T_1^*, T, T_2^* is a fan of M and (iii) of Lemma 4.3 holds, or T_1^*, T, T_2^* is not a fan of M. We may therefore assume that the latter possibility holds. Then, as $|E(N)| \geq 4$ and so $|E(M)| \geq 7$, there is a triangle T' of M such that T', T_1^*, T, T_2^* or T_1^*, T, T_2^*, T' is a chain of M. The first case implies that $a \in T'$ and the second that $a'' \in T'$. But M has no triangle containing a so $a'' \in T'$. Hence $T' \cap (T_1^* \cup T \cup T_2^*) = \{y, a''\}$. But $N = M/a, y\backslash x$ and so N has a circuit of size at most two containing a''; a contradiction. This completes the proof of Lemma 4.3. □

4.7. Lemma. *Suppose that (M, A) is a minimal pair in which $|A| \geq 7$. Let e be an element of $E(M) - \mathrm{cl}(A)$ that is in exactly h triads, $T_1^*, T_2^*, \ldots, T_h^*$, for some $h \geq 2$. Suppose that $T_i^* \cap A = \{a_i, b_i\}$ for all i. Then (M', A') is a minimal pair where $M' = M\backslash e/\{a_1, a_2, \ldots, a_h\}$ and $A' = A - \{a_1, a_2, \ldots, a_h\}$.*

Proof. We note first that, by (vi) of Lemma 4.1, if $i \neq j$, then $T_i^* \cap T_j^* = \{e\}$. Moreover, by (i) of Lemma 4.1, M' is 3-connected. Now M' has a 3-connected minor N' such that (N', A') is a minimal pair and $N'|A' = M'|A'$. Then $N' = M'\backslash X/Y$ for some independent set Y and coindependent set X. Clearly

4.7.1. *$X \cup e$ is coindependent in M.*

We shall complete the proof of the lemma by showing that both X and Y are empty. First we show that

4.7.2. $r(A) + r(Y) = r(A \cup Y)$.

By orthogonality, $\{a_1, a_2, \ldots, a_h\}$ is independent in $M\backslash e$ and hence in M. As Y is independent in $M/\{a_1, a_2, \ldots, a_h\}$, it follows that $\{a_1, a_2, \ldots, a_h\} \cup Y$ is independent in M. Thus, as $N'|A' = M'|A'$, we have

$$r_{M/\{a_1,a_2,\ldots,a_h\}/Y}(A - \{a_1, a_2, \ldots, a_h\}) = r_{M/\{a_1,a_2,\ldots,a_h\}}(A - \{a_1, a_2, \ldots, a_h\}).$$

Therefore

$$r(A \cup Y) - r(\{a_1, a_2, \ldots, a_h\} \cup Y) = r(A) - r(\{a_1, a_2, \ldots, a_h\}),$$

so $r(A \cup Y) - h - |Y| = r(A) - h$, and (4.7.2) follows.

Next we observe that

4.7.3. $|E(N')| \geq 4$.

To see this, we note that, since $|A| \geq 7$ and

$$E(N') \supseteq A - \{a_1, a_2, \ldots, a_h\} \supseteq \{b_1, b_2, \ldots, b_h\},$$

it follows that $|A| - h \geq h$. Thus, if $h \leq 3$, then $|A| - h \geq 4$, and (4.7.3) holds; and if $h \geq 4$, then $|E(N')| \geq h \geq 4$, and again (4.7.3) holds.

Now $M\backslash(X \cup e)/Y$ is obtained from the 3-connected matroid N' by adding a_i in series with b_i for all i. Hence $M\backslash(X \cup e)/Y$ is connected. Moreover, the following is an easy consequence of the fact that $|E(N')| \geq 4$.

4.7.4. *Every 2-separation of $M\backslash(X \cup e)/Y$ has some $\{a_i, b_i\}$ as a part.*

We show next that

4.7.5. *T_i^* is a triad of $M\backslash X/Y$ for every i.*

Suppose that this fails for some i. Then, since T_i^* is a triad of M and $T_i^* - e$ is a cocircuit of $(M\backslash X/Y)\backslash e$, it follows that either (i) T_i^* is contained in a series class of $M\backslash X/Y$; or (ii) e is a coloop of $M\backslash X/Y$. But, since $X \cup e$ is coindependent in M, (ii) cannot occur. Likewise, (i) cannot occur, otherwise $\{b_i, e\}$ is a cocircuit of

$M\backslash X/Y$, so b_i is a coloop of $(M\backslash X/Y)\backslash e$ and hence of N'; a contradiction. We conclude that (4.7.5) holds.

Next we shall prove that

4.7.6. *$M\backslash X/Y$ is 3-connected.*

Since $(M\backslash X/Y)\backslash e$ is connected and e is in a triad of $M\backslash X/Y$, the last matroid is certainly connected. Assume that (4.7.6) fails and let $\{Z, W\}$ be a partition of $E((M\backslash X/Y)\backslash e)$ such that $\{Z \cup e, W\}$ is a 2-separation of $M\backslash X/Y$. Then $\{Z, W\}$ is a 2-separation of $(M\backslash X/Y)\backslash e$ unless $|Z| = 1$. But, in the exceptional case, $Z \cup e$ is either (i) a 2-element cocircuit, or (ii) a 2-element circuit of $M\backslash X/Y$. The first case implies the contradiction that Z is a 1-element cocircuit of the connected matroid $(M\backslash X/Y)\backslash e$. In the second case, orthogonality and (4.7.5) imply that Z meets the h disjoint sets of the form $T_i^* - e$. This is a contradiction since $|Z| = 1$ and $h \geq 2$. We conclude that $\{Z, W\}$ is indeed a 2-separation of $(M\backslash X/Y)\backslash e$.

By (4.7.4), one of Z and W is $\{a_i, b_i\}$ for some i. Suppose first that $Z = \{a_i, b_i\}$. Then $Z \cup e = T_i^*$ and, in $M\backslash X/Y$, we have

$$1 = r(Z \cup e) + r(W) - r(M\backslash X/Y) = r(Z \cup e) + r^*(Z \cup e) - |Z \cup e| = r(Z \cup e) + 2 - 3.$$

Hence $r(Z \cup e) = 2$. But $r(Z) = r(\{a_i, b_i\}) = 2$ since a_i and b_i are in series in $(M\backslash X/Y)\backslash e$. Thus e is in a circuit C of $M\backslash X/Y$ contained in T_i^*. This contradicts orthogonality since, for $j \neq i$, the triad T_j^* of $M\backslash X/Y$ meets C in a single element. We conclude that $Z \neq \{a_i, b_i\}$.

We may now assume that $W = \{a_i, b_i\}$. Since $\{Z \cup e, W\}$ is a 2-separation of $M\backslash X/Y$, it follows that $\{a_i, b_i\}$ is a circuit or a cocircuit of $M\backslash X/Y$. But $\{a_i, b_i\}$ is a proper subset of the triad T_i^* of $M\backslash X/Y$, so $\{a_i, b_i\}$ is certainly not a cocircuit of $M\backslash X/Y$. Moreover, if $\{a_i, b_i\}$ is a circuit of $M\backslash X/Y$, then b_i is a loop of $(M\backslash X/Y)\backslash e/\{a_1, a_2, \ldots, a_h\}$, that is, of N'. This contradiction completes the proof of (4.7.6).

By (4.7.2), $r(A \cup Y) = r(A) + r(Y)$, so $(M\backslash X/Y)|A = M|A$. Since (M, A) is a minimal pair, it follows that $X = Y = \emptyset$. Hence $N' = M'$ and so (M', A') is a minimal pair. □

5. Proofs of the main theorems

Theorems 1.5 and 1.6 can be combined into the following result.

5.1. Theorem. *If (M, A) is a minimal pair, then*

$$|E(M)| \leq |A| + \lambda_1(A, M) + \lambda_2(A, M) + \min\{\lambda_3(A, M), r(M) - r_M(A)\} - \alpha(A, M),$$

where

$$\alpha(A, M) = \begin{cases} 0, & \text{when } A \text{ is a circuit of } M \\ 1, & \text{when } A \text{ is not a circuit} \end{cases}$$

This theorem will be derived from two propositions. The first of these, which is stated and proved below, gives the structure of a minimal counterexample to the theorem; the second, which was proved in an earlier paper [6], will be stated following the proof of the first.

5.2. Proposition. *Let (M, A) be a minimal pair that is a counterexample to Theorem 5.1 and is chosen so that*

$$(|E(M)|, -|A|)$$

is lexicographically minimal among such counterexamples. Then every element of $E(M)-\mathrm{cl}(A)$ *belongs to some type-2 fan of length three in which the rim is contained in a 4-circuit of* $M|A$ *and the spokes are contained in* $E(M)-\mathrm{cl}(A)$.

Proof. Suppose that (M, A) does not satisfy the proposition. Then, clearly,

$$E(M) - \mathrm{cl}(A) \neq \emptyset. \tag{8}$$

Moreover, by (iii) of Lemma 4.1,

$$|A| \geq 3. \tag{9}$$

The rest of the proof of the proposition will be broken up into a sequence of lemmas, the first of which is as follows.

5.3. Lemma. *M is neither a wheel having rim $\mathrm{cl}(A)$ nor a whirl having rim A.*

Proof. Suppose that M is the rank-n whirl having rim A. Then A is independent and

$$\begin{aligned}&|A| + \lambda_1(A, M) + \lambda_2(A, M)\\&+ \min\{\lambda_3(A, M), r(M) - r_M(A)\} - \alpha(A, M) = 3|A| - 1 = 3n - 1.\end{aligned}$$

But (M, A) is a counterexample to Theorem 5.1 and $|E(M)| = 2n$, so $2n > 3n - 1$; a contradiction.

Next suppose that M is the rank-n wheel having rim $\mathrm{cl}(A)$ but $A \neq \mathrm{cl}(A)$. Then $A = \mathrm{cl}(A) - a$ for some element a, so A is independent. Again, we have that

$$\begin{aligned}&|A| + \lambda_1(A, M) + \lambda_2(A, M)\\&+ \min\{\lambda_3(A, M), r(M) - r_M(A)\} - \alpha(A, M) = 3|A| - 1 = 3n - 4.\end{aligned}$$

But $|E(M)| = 2n$ and (M, A) is a counterexample to Theorem 5.1. Thus $2n > 3n-4$ so $n \leq 3$. Hence $|A| = n - 1 \leq 2$; a contradiction to (9).

Finally, suppose that M is the rank-n wheel and A is its rim. In this case, $n \geq 4$, otherwise $n = 3$ and A is a 3-element circuit so $(M|A, A)$ is a minimal pair; a contradiction. Now $\lambda_1(A, M) = 1, \lambda_2(A, M) = n - 2, \lambda_3(A, M) = n - 3, r(M) - r(A) = 1$, and $\alpha(A, M) = 0$. Thus

$$2n = |E(M)| > n + 1 + (n - 2) + 1 - 0.$$

This contradiction completes the proof of Lemma 5.3. □

In several of the lemmas below, we shall replace the minimal pair (M, A) by another minimal pair (M', A') such that $(|E(M')|, -|A'|)$ is lexicographically less than $(|E(M)|, -|A|)$. Thus (M', A') must satisfy the conclusion of Theorem 5.1 although (M, A) does not. Hence

$$\begin{aligned}&|A| + \lambda_1(A, M) + \lambda_2(A, M)\\&+ \min\{\lambda_3(A, M), r(M) - r_M(A)\} - \alpha(A, M) - |E(M)| < 0\end{aligned}$$

and

$$\begin{aligned}&|A'| + \lambda_1(A', M') + \lambda_2(A', M')\\&+ \min\{\lambda_3(A', M'), r(M') - r_{M'}(A')\} - \alpha(A', M') - |E(M')| \geq 0.\end{aligned}$$

On taking the difference of these two inequalities, we get

$$\delta_A + \delta_1 + \delta_2 + \delta_m - \delta_\alpha - \delta_E < 0, \tag{10}$$

where

$$
\begin{aligned}
\delta_A &= |A| - |A'|, \\
\delta_1 &= \lambda_1(A, M) - \lambda_1(A', M'), \\
\delta_2 &= \lambda_2(A, M) - \lambda_2(A', M'), \\
\delta_m &= \min\{\lambda_3(A, M), r(M) - r_M(A)\} - \min\{\lambda_3(A', M'), r(M') - r_{M'}(A')\}, \\
\delta_\alpha &= \alpha(A, M) - \alpha(A', M'), \\
\delta_E &= |E(M)| - |E(M')|.
\end{aligned}
$$

Hence we shall arrive at a contradiction whenever (10) is not satisfied. To deal with δ_m, we define two other differences:

$$
\begin{aligned}
\delta_3 &= \lambda_3(A, M) - \lambda_3(A', M'), \\
\delta_r &= [r(M) - r_M(A)] - [r(M') - r_{M'}(A')].
\end{aligned}
$$

The elementary proof of the next lemma is omitted.

5.4. Lemma. (i) *If δ_r or δ_3 is zero, then $\delta_m \geq \min\{\delta_r, \delta_3\}$.*
(ii) *If $\delta_r = \delta_3$, then $\delta_m = \delta_r = \delta_3$.*

5.5. Lemma. *Every triangle of M that meets $E(M) - \mathrm{cl}(A)$ contains an element of $\mathrm{cl}(A)$.*

Proof. Suppose that T is a triangle of M that avoids $\mathrm{cl}(A)$. By (viii) of Lemma 4.1, there is a triad that meets T in exactly two elements and contains some element a of A. Then, by Lemma 4.2, if $M' = M/T$ and $A' = A - a$, then (M', A') is a minimal pair and

$$(M/T)|(A - a) = M|(A - a).$$

As $|E(M')| < |E(M)|$, the minimal pair (M', A') satisfies Theorem 5.1. Hence (10) holds. We shall now consider each individual difference in (10). Clearly

$$\delta_A = 1 \text{ and } \delta_E = 3.$$

Next we note that, since $(M/T)|(A - a) = M|(A - a)$ and a is a coloop of $M|A$, we have

$$\delta_1 = \delta_2 = 1 \text{ and } \delta_3 = 0.$$

Now $r(M') = r(M) - 2$ and $r_{M'}(A') = r_M(A - a) = r_M(A) - 1$. It follows that $\delta_r = 1$ and hence, by Lemma 5.4,

$$\delta_m \geq 0.$$

Finally, (viii) of Lemma 4.1 implies that $M|A$ has at least three coloops including a, and therefore

$$\delta_\alpha = 0.$$

On combining the above differences, we have

$$\delta_A + \delta_1 + \delta_2 + \delta_m - \delta_\alpha - \delta_E \geq 1 + 1 + 1 + 0 - 0 - 3 = 0.$$

This contradiction to (10) completes the proof of the lemma. □

5.6. Lemma. *If T^* is a triad of M such that $|T^* \cap \mathrm{cl}(A)| \geq 2$, then $|T^* \cap A| \geq 2$.*

Proof. Suppose that $|T^* \cap A| \leq 1$. By Lemma 2.5, $T^* \cap A = \{a\}$ for some a. Choose e in $(T^* \cap \mathrm{cl}(A)) - A$. Let $M' = M$ and $A' = A \cup e$. Clearly (M', A') is a minimal pair. Observe that the connected component N_e of $M|(A \cup e)$ that contains e must also contain a. Moreover, a is a coloop of $N_e \backslash e$. Let l be the number of coloops of $N_e \backslash e$ that are not coloops of N_e. Then $l \geq 1$. By Lemma 2.9(i) of [6], we have that

$$\delta_1 + \delta_2 \geq l.$$

Observe also that

$$\delta_A = -1, \delta_E = 0, \text{ and } \delta_r = 0,$$

where the last of these holds because A spans e.

Now consider a series class S of $M|A$. Its contribution to $\lambda_3(A, M)$ is $\max\{0, |S| - 3\}$. In $M|(A \cup e)$, there is a partition $\{S_1, S_2, \ldots, S_k\}$ of S such that each S_i is a series class of $M|(A \cup e)$. The total contribution of these series classes to $\lambda_3(A \cup e, M)$ is

$$\sum_{i=1}^{k} \max\{0, |S_i| - 3\},$$

which clearly does not exceed $\max\{0, |S| - 3\}$. The only non-trivial series class of $M|(A \cup e)$ that is not contained in a non-trivial series class of $M|A$ is the one that contains e and the set of l elements that are coloops of $N_e \backslash e$ but not of N_e. Its contribution to $\lambda_3(A \cup e, M)$ is $\max\{0, l - 2\}$. We conclude that

$$\delta_3 \geq -\max\{0, l - 2\} = \min\{0, 2 - l\}$$

and, since $\delta_r = 0$, Lemma 5.4 implies that

$$\delta_m \geq \min\{0, 2 - l\}.$$

Thus

$$\delta_A + (\delta_1 + \delta_2) + \delta_m - \delta_E \geq -1 + l + \min\{0, 2 - l\} - 0.$$

By (10), we must have that $\delta_\alpha > l - 1 + \min\{0, 2 - l\}$. Hence

$$\delta_\alpha > \min\{l - 1, 1\}.$$

But $\delta_\alpha \leq 1$ and $l \geq 1$, so $\delta_\alpha = 1$ and $l - 1 = 0$. The first of these implies that $A \cup e$ is a circuit of M. Hence $N_e = M|(A \cup e)$ and A equals the set of coloops of $N_e \backslash e$ that are not coloops of N_e. Thus $l = |A| \geq 3$, by (9); a contradiction. $\square$

The proof of Lemma 5.8 will use the following result of Oxley and Wu [11, Lemma 2.4].

5.7. Lemma. *For some $n \geq 2$, let $\{e_1, e_2, \ldots, e_n\}$ and $\{f_1, f_2, \ldots, f_n\}$ be disjoint subsets of the ground set of a 3-connected matroid N. Suppose that, for all i in $\{1, 2, \ldots, n-1\}$ and all j in $\{1, 2, \ldots, n\}$, $\{e_i, f_i, e_{i+1}\}$ is a triangle and $\{f_j, e_{j+1}, f_{j+1}\}$ is a triad, where all subscripts are read modulo n. Then N is isomorphic to a wheel or whirl of rank n.*

5.8. Lemma. *Let f be an element of $E(M) - \mathrm{cl}(A)$. Suppose that either*

(i) *f is in a chain with at least three links such that all the spokes of this chain are in $E(M) - \mathrm{cl}(A)$ and the rim of this chain is contained in $\mathrm{cl}(A)$; or*
(ii) *f is in a unique triad of M but is in no triangles.*

Then f is in a type-2 fan of length three in which the rim is a series class of $M|A$ and the spokes are contained in $E(M) - \mathrm{cl}(A)$.

Proof. We shall show first that, in each case, there are elements a of A and e of $E(M) - \mathrm{cl}(A)$ such that $M/a\backslash e$ has a minimal 3-connected minor M' for which $M'|A' = (M/a)|A'$ where $A' = A - a$ and either

(a) $M' = M/a\backslash e$ where $\{a, e\}$ is contained in a triad of M; or
(b) $M' = M/a\backslash e/e'$ for some element e' of $E(M) - \mathrm{cl}(A)$ such that $f \in \{e, e'\}$. Moreover, for some elements a_1 and a_2 of A, the set $\{e, a_1, e'\}$ is a triangle, and $\{a, e, a_1\}, \{e, a_1, e'\}, \{a_1, e', a_2\}$ is a fan of M.

Suppose that (ii) occurs and let T^* be a triad containing f. Then $T^* - f$ meets A, otherwise, by Lemma 4.1(v), $|T^* - \mathrm{cl}(A)| \leq 1$ in which case, by Lemma 5.6, $|T^* \cap A| \geq 2$; a contradiction. Let $T^* - f = \{a, g\}$ where $a \in A$. Now suppose that a is in a triangle of M. Then, since this triangle cannot contain f, it must contain g. But, in that case, the cosimplification of $M\backslash f$ is not 3-connected, contradicting Lemma 4.1(i). We conclude that a is in no triangles of M.

We are aiming to apply Lemma 4.3 to M. We know, by Lemmas 5.5 and 5.6, that hypotheses (a) and (b) of that lemma hold and that a is in no triangles of M. Now consider M/a. It must be 3-connected otherwise neither M/a nor M/f is 3-connected and the dual of Tutte's triangle lemma [9, Lemma 8.4.9] implies that a is in a triangle; a contradiction. Next consider $M\backslash f$. Since f is in no triangles, the simplification of M/f equals M/f and so is not 3-connected. Thus the cosimplification of $M\backslash f$ is 3-connected. Since T is the unique triad of M containing f, this cosimplification is $M\backslash f/a$. Thus $M/a\backslash f$ is 3-connected. Therefore there is a minimal 3-connected minor M' of M/a for which $M'|(A - a) = (M/a)|(A - a)$ such that M' is a minor of $M/a\backslash f$. Then Lemma 4.3 and the fact that f is in no triangles imply that $M' = (M/a)\backslash f$. We conclude that if (ii) occurs, then (a) holds with $e = f$.

Next suppose that (i) holds, and take a maximal such chain $T_1, T_2, \dots, T_k$ of M. Then $k \geq 3$. Assume that there is a set T_{k+1} such that $T_2, T_3, \dots, T_{k+1}$ is a chain. Let $T_i = \{e_i, e_{i+1}, e_{i+2}\}$ for all i. Suppose first that $T_1, T_2, \dots, T_{k+1}$ is a chain. By the choice of $T_1, T_2, \dots, T_k$, we deduce that either e_{k+3} is a spoke of $T_1, T_2, \dots, T_{k+1}$ that is in $\mathrm{cl}(A)$, or e_{k+3} is a rim element of $T_1, T_2, \dots, T_{k+1}$ that is not in $\mathrm{cl}(A)$. In the first case, T_{k+1} is a triangle containing two elements of $\mathrm{cl}(A)$, so $e_{k+1} \in \mathrm{cl}(A)$; a contradiction. In the second case, by Lemmas 5.5 and 4.1(xi), e_{k+2} belongs to a triad T_0^* that meets $\mathrm{cl}(A)$ in two elements. By orthogonality with T_k, we must have that $e_{k+1} \in T_0^*$, so, by Lemma 2.8, $T_0^* = T_{k+1}$; a contradiction. We conclude that $T_1, T_2, \dots, T_{k+1}$ is not a chain. Then $e_{k+3} \in T_1 \cup T_2 \cup \ldots \cup T_k$ and it follows, by orthogonality, that $e_{k+3} = e_1$ and that T_1 and T_{k+1} are either both triangles or are both triads. Then, by Lemma 5.7, it follows that M is isomorphic to a wheel or whirl of rank $k+2$. Moreover, clearly this wheel or whirl has rim $\mathrm{cl}(A)$ and so we have a contradiction to Lemma 5.3. We conclude that there is no set T_{k+1} such that $T_2, T_3, \dots, T_{k+1}$ is a chain. Similarly, there is no set T_0 such that $T_0, T_1, \dots, T_{k-1}$ is a chain. We conclude that $T_1, T_2, \dots, T_k$ is a fan F. Moreover, since e_1 and e_{k+2} are non-essential, it follows by Lemma 4.1(ix) that both T_1 and T_k are triads of M.

Let $\{a, e, a_1\}$ be the first link T^* of F where a is the rim element of T^* that is in no triangles, and e is the spoke of T^*, the first spoke of F. By Lemma 5.6, $\{a, a_1\} \subseteq A$. By Lemma 4.1(i), the cosimplification of $M\backslash e$ is 3-connected. Suppose that this cosimplification is isomorphic to $M\backslash e/a$, and let M' be a minimal 3-connected minor of $M\backslash e/a$ for which $M'|(A - a) = (M/a)|(A - a)$. Then Lemma 4.3 implies that (a) or (b) holds provided that, as we now show, $f \in \{e, e'\}$. Clearly e is in both the fan F and the fan $\{a, e, a_1\}, \{e, a_1, e'\}, \{a_1, e', a_2\}$ whose existence is asserted in Lemma 4.3(iii). By Theorem 2.2, if these fans are distinct, then F has five elements and has e as a rim element; a contradiction. Hence these two fans are the same, so $f \in \{e, e'\}$. We conclude that if the cosimplification of $M\backslash e$ is isomorphic to $M\backslash e/a$, then (a) or (b) holds.

Now assume that the cosimplification of $M\backslash e$ is not isomorphic to $M\backslash e/a$. Then M has a triad T_1^* that contains e and is different from T^*. It follows, using orthogonality, that T_1^* must contain the first two spokes, e and e', of F. Moreover, by Lemma 4.1(v), the other element a' of T_1^* is in A. By orthogonality again, F cannot have a third spoke. Hence e and e' are the only two spokes of F, so $f \in \{e, e'\}$. Moreover, T^* and T_1^* are the only triads of M containing e, and the cosimplification of $M\backslash e$ is isomorphic to $M\backslash e/a/e'$. We show next that

$$(M\backslash e/a/e')|(A - a) = (M/a)|(A - a).$$

This follows if $(M/a)|[(A-a) \cup e']$ has e' as a coloop. Assume the contrary. Then e' is in the closure of $A - a$ in M/a, so e' is in the closure of A in M; a contradiction. Clearly $M\backslash e/a/e'$ has a minimal 3-connected minor M' for which $M'|(A - a) = (M/a)|(A-a)$. Thus, by Lemma 4.3, we have that (a) or (b), and hence (b), holds.

We now know that (a) or (b) holds, and we examine the minimal pair (M', A'). Since $|E(M')| < |E(M)|$, Theorem 5.1 holds for (M', A'). Clearly

$$\delta_A = 1.$$

Moreover,

$$\delta_r = \begin{cases} 0, & \text{when } M' = M/a\backslash e; \text{ and} \\ 1, & \text{when } M' = M/a\backslash e/e'. \end{cases}$$

The rest of the argument will be broken into the following cases.

(I) a is a coloop of $M|A$;

(II) a is in a series class of $M|A$.

In case (I), since $M'|A' = (M/a)|A'$,

$$\delta_1 = 1, \delta_2 = 1, \text{ and } \delta_3 = 0.$$

Thus, by Lemma 5.4, $\delta_m \geq 0$. Hence, by (10),

$$1 + 1 + 1 + 0 \leq \delta_A + \delta_1 + \delta_2 + \delta_m < \delta_\alpha + \delta_E.$$

Thus, as $\delta_\alpha \leq 1$ and $\delta_E \leq 3$, we deduce that $\delta_\alpha = 1$ and $\delta_E = 3$. The first of these implies that A is not a circuit of M but $A - a$ is a circuit of M'; and the second implies that (b) holds. By (b), $M\backslash e$ has $\{a, a_1\}$ as a cocircuit. Thus $M|A$ has $\{a, a_1\}$ as a union of cocircuits. But a is a coloop of $M|A$. Hence so is a_1. This contradicts the fact that $A - a$ is a circuit of $(M|A)/a$. Thus (I) cannot occur.

Now consider (II). In this case, $M|A$ is obtained from $M'|A'$ by adding a in series to some element of A'. Thus $M|A$ is isomorphic to the 2-sum of $M'|A'$ and a copy of $U_{2,3}$. Hence

$$\delta_1 = 0, \delta_2 = 1, \text{ and } \delta_\alpha = 0.$$

Moreover, if S_a is the series class of $M|A$ containing a, then

$$\delta_3 = \begin{cases} 0, & \text{when } |S_a| \leq 3; \\ 1, & \text{when } |S_a| \geq 4. \end{cases}$$

Since $\delta_r \in \{0,1\}$, it follows, by Lemma 5.4, that

$$\delta_m \geq 0.$$

Thus, by (10),

$$1+0+1+0-0 \leq 1+0+1+\delta_m - 0 = \delta_A + \delta_1 + \delta_2 + \delta_m - \delta_\alpha < \delta_E.$$

Since $\delta_E \leq 3$, we deduce that $\delta_E = 3$ and $\delta_m = 0$. Hence (b) holds, that is, $M' = M/a\backslash e/e'$, so $\delta_r = 1$. Thus, by Lemma 5.4, $\delta_3 \neq 1$ otherwise $\delta_m = 1$. Hence $\delta_3 = 0$ so $|S_a| \leq 3$. Therefore, by (b), $S_a = \{a, a_1, a_2\}$ and the fan $\{a, e, a_1\}, \{e, a_1, e'\}, \{a_1, e', a_2\}$ satisfies all the conditions asserted in the lemma. □

5.9. Lemma. $|A| \geq 7$.

Proof. As $E(M) - \mathrm{cl}(A) \neq \emptyset$, there is a cocircuit D of M that avoids $\mathrm{cl}(A)$. Moreover, since (M, A) is a minimal pair, for all elements e of D, the matroid M/e is not 3-connected. Thus, by Theorem 2.7, M has two triangles T_1 and T_2 meeting D in distinct subsets. By Lemma 5.5, each T_i meets $\mathrm{cl}(A)$. Hence, by Lemma 4.1(ix), there is a unique element a_{T_i} in $T_i \cap \mathrm{cl}(A)$ and there is a chain $T^*_{1T_i}, T_i, T^*_{2T_i}$ whose rim A_{T_i} is contained in $\mathrm{cl}(A)$. Take an element f_1 of $(T_1 - T_2) - \mathrm{cl}(A)$. By Lemma 5.8, f_1 is in a 5-element fan F'_1 whose rim is a series class of $M|A$ and all of whose spokes are in $E(M) - \mathrm{cl}(A)$. Clearly $T_2 - \mathrm{cl}(A)$ has an element f_2 that is not in F'_1 and, by Lemma 5.8 again, f_2 is in a 5-element fan F'_2 whose rim is a series class of $M|A$.

The rims R_1 and R_2 of F'_1 and F'_2 are both series classes of $M|A$ and, by Theorem 2.2, these rims are not equal. Hence R_1 and R_2 are disjoint and so $|A| \geq 6$. We may now assume that $|A| = 6$ otherwise the lemma holds. But, in that case, $M|A$ is the union of the two series classes R_1 and R_2. Hence each R_i is a triangle of $M|A$. This is a contradiction as F'_i is a fan. □

5.10. Lemma. *An element e of $E(M) - \mathrm{cl}(A)$ that does not belong to a triangle of M belongs to exactly one triad.*

Proof. The element e certainly belongs to at least one triad, otherwise, by Lemma 2.3, either $M\backslash e$ or M/e is 3-connected; a contradiction to the fact that (M, A) is a minimal pair. Now let $T^*_1, T^*_2, \ldots, T^*_n$ be the triads of M that contain e and suppose that $n \geq 2$. For all i, we must have that $|T^*_i - \mathrm{cl}(A)| < 2$, otherwise Tutte's triangle lemma implies that e is in a triangle. Hence $|T^*_i \cap \mathrm{cl}(A)| \geq 2$ for all i and it follows, by Lemma 5.5, that $|T^*_i \cap A| = 2$. Let $T^*_i \cap A = \{a_i, b_i\}$ for all i. Then, by Lemma 4.7, (M', A') is a minimal pair where $M' = M\backslash e/\{a_1, a_2, \ldots, a_n\}$ and $A' = A - \{a_1, a_2, \ldots, a_n\}$.

Clearly

$$\delta_A = n \text{ and } \delta_E = n+1.$$

Moreover, for each i, either a_i is in series with b_i in $M|A$, or a_i is a coloop of $M|A$. Without loss of generality, assume that the set of coloops of $M|A$ in $\{a_1, a_2, \ldots, a_n\}$ is $\{a_1, a_2, \ldots, a_k\}$. Then a matroid isomorphic to $M|A$ can be obtained from $M'|A'$ in two stages: first take the direct sum of $M'|A'$ with the k coloops $a_1, a_2, \ldots, a_k$; and then take the 2-sum of the resulting matroid with $n-k$ copies of $U_{2,3}$ using $b_{k+1}, b_{k+2}, \ldots, b_n$ as basepoints. Thus

$$\delta_1 = k \geq 0 \text{ and } \delta_2 = n.$$

As each series class of $M'|A'$ is contained in a series class of $M|A$, we have $\delta_3 \geq 0$. Moreover, as δ_r is clearly zero, Lemma 5.4 implies that

$$\delta_m \geq 0.$$

Therefore, by (10), we obtain a contradiction unless

$$n + 0 + n + 0 \leq \delta_A + \delta_1 + \delta_2 + \delta_m < \delta_E + \delta_\alpha = (n + 1) + \delta_\alpha.$$

Thus $1 \leq n - 1 < \delta_\alpha \leq 1$; a contradiction. □

On combining Lemmas 5.8 and 5.10, we obtain the following.

5.11. Corollary. *If $f \in E(M) - \mathrm{cl}(A)$, then f is in a type-2 fan of length three in which the rim is a series class of $M|A$ and the spokes are contained in $E(M) - \mathrm{cl}(A)$. Moreover, two distinct such fans are disjoint.*

Proof. Either f is in a triangle of M or not. In the first case, by Lemma 5.10, f is in a unique triad but no triangle and the first part of the corollary follows by Lemma 5.8. In the second case, by Lemma 5.5, f is in a triangle with some element of $\mathrm{cl}(A)$. Then, by Lemma 4.1(ix), f satisfies (i) of Lemma 5.8 and again the first part of the corollary follows by that lemma.

Now observe that the intersection of two distinct fans F_1 and F_2 satisfying the specified conditions is a subset of the intersection of their rims or the intersection of their sets of spokes. But each rim is a series class of $M|A$ so the rims are either equal or disjoint. Thus if F_1 meets F_2, then F_1 and F_2 have a common element that is essential in M. But this violates Theorem 2.2. □

Now let T_0, T_1, T_2 be a type-2 fan F of M where $T_0 = \{a_0, e_1, a_1\}, T_1 = \{e_1, a_1, e_2\}$, and $T_2 = \{a_1, e_2, a_2\}$ where $\{a_0, a_1, a_2\} \subseteq \mathrm{cl}(A)$ and $\{e_1, e_2\} \cap \mathrm{cl}(A) = \emptyset$. A *chord* of F is an element y such that $R \cup y$ is a circuit of M where R is the rim $\{a_0, a_1, a_2\}$ of F. In view of the last corollary, to complete the proof of Proposition 5.2, it suffices to show that F has a chord that is contained in A.

5.12. Lemma. *If y is a chord of F, then y is unique and $y \in A$.*

Proof. If $\{y, a_0, a_1, a_2\}$ and $\{y', a_0, a_1, a_2\}$ are circuits of M, then $\{y, y', a_0, a_2\}$ contains a circuit, which, by orthogonality, must be $\{y, y'\}$; a contradiction. Thus y is unique. Now assume that $y \notin A$. Then $M\backslash y$ is not 3-connected. Let $\{X, Y\}$ be a 2-separation of $M\backslash y$ and suppose, without loss of generality, that $|X \cap T_1| \geq 2$. Subject to this restriction, choose the 2-separation $\{X, Y\}$ such that $|X \cap F|$ is maximum.

We now show that $|Y| \geq 3$. Assume that $|Y| = 2$. As M is simple, Y must be a cocircuit of $M\backslash y$, and so $Y \cup y$ is a triad T^* of M. By orthogonality, $T^* \cap R \neq \emptyset$. By Lemma 2.8, $a_1 \notin T^*$. Hence we may assume that $a_0 \in T^*$. Because R is a series class of $M|A$, the unique element, z say, of $T^* - \{y, a_0\}$ must belong to A otherwise a_0 is a coloop of $M|A$. Thus z and a_0 are in series in $M|A$ so $z = a_2$, and $T^* = \{y, a_0, a_2\}$. By elimination and orthogonality, $\{a_0, e_1, e_2, a_2\}$ is a cocircuit of M. Hence $(\{a_0, e_1, e_2, a_2\} \cup \{y, a_0, a_2\}) - a_0$ contains a cocircuit D of M. But $M|A$ has a circuit containing $\{a_0, a_1, a_2\}$ so, by orthogonality, $D = \{e_1, e_2, y\}$. This is a contradiction since $|D \cap (R \cup y)| = 1$. We conclude that $|Y| \geq 3$.

Next we observe that, since X spans T_1, it follows that $\{X \cup T_1, Y - T_1\}$ is a 2-separation of $M\backslash y$. By the choice of $\{X, Y\}$, we deduce that $X \cup T_1 = X$, so $T_1 \subseteq X$. Thus X spans T_0 in M^*, so $\{X \cup T_0, Y - T_0\}$ is a 2-separation of $M\backslash y$, which, as above, must equal $\{X, Y\}$. Thus $T_0 \subseteq X$ and, by symmetry, $T_2 \subseteq X$. We conclude that $F \subseteq X$, so X spans y. This contradiction completes the proof that $y \in A$. □

We may now assume that F has no chord otherwise the result follows.

5.13. Lemma. *$M/\{a_0, a_2\}$ is 3-connected.*

Proof. As $a_0 \in T_0 - T_1$ and T_0, T_1, T_2 is a type-2 fan, a_0 is non-essential so M/a_0 is 3-connected. We show next that M/a_0 has no triangle containing a_2. Assume that such a triangle T' exists.

Suppose first that $e_2 \in T'$. Then $T' = \{a_2, e_2, e_3\}$ for some e_3, which, since $e_2 \notin \mathrm{cl}(A)$, is not in $\mathrm{cl}(A)$. Thus either T' or $T' \cup a_0$ is a circuit of M. The first possibility contradicts the fact that T_0, T_1, T_2 is a fan; the second contradicts orthogonality with the triad T_0 since $e_3 \neq e_1$ otherwise M/a_0 has a line containing $\{a_1, a_2, e_1, e_2\}$.

We may now assume that $e_2 \notin T'$. Then, by orthogonality with T_2, we have $T' = \{a_2, a_1, g\}$ for some element g. By Lemma 2.8, T' is not a triangle of M so $T' \cup a_0$ is a circuit of M and hence g is a chord of M; a contradiction.

We conclude that there is indeed no triangle of M/a_0 containing a_2. But the cosimplification of $M/a_0\backslash a_2$ is not 3-connected, so the simplification of $M/a_0/a_2$, which equals $M/\{a_0, a_2\}$, is 3-connected. □

Now let $A' = A - \{a_0, a_2\}$ and let M' be a 3-connected matroid such that $M'|A' = (M/\{a_0, a_2\})|A'$ and (M', A') is a minimal pair. Then $M' = (M/\{a_0, a_2\})\backslash X/Y$ for some sets X and Y.

5.14. Lemma. *$Y \cup (X - \mathrm{cl}(A)) \subseteq \{e_1, e_2\}$.*

Proof. Suppose that there is an element z in $[Y \cup (X - \mathrm{cl}(A))] - \{e_1, e_2\}$. We shall show that $z \notin \mathrm{cl}(A)$. If not, then $z \in Y \cap \mathrm{cl}(A)$ and

$$r(M'|A') \leq r((M/\{a_0, a_2, z\})|A') = r((M/\{a_0, a_2\})|A') - 1;$$

a contradiction. We conclude that we do indeed have that $z \notin \mathrm{cl}(A)$. Thus $z \in E(M) - (\mathrm{cl}(A) \cup F)$. Then, by Corollary 5.11, z is in a type-2 fan F' of M of length three whose rim is a series class of $M|A$ and such that $F' \subseteq E(M) - F$. But $M\backslash z$ has a 2-cocircuit contained in $A - \{a_0, a_2\}$ that remains a 2-cocircuit of $M/\{a_0, a_2\}\backslash X/Y$. Now the last matroid is 3-connected of size at least five since $|A| \geq 7$. Hence $z \notin X$. On the other hand, if $z \in Y$, then M/z has a 2-circuit containing a spoke z' of F'. Thus M' is a minor of $M/z\backslash z'$ and hence of $M\backslash z'$. Again we obtain a contradiction. □

5.15. Lemma. *$X \cap \mathrm{cl}(A) = \emptyset$.*

Proof. Suppose that $X \cap \mathrm{cl}(A) \neq \emptyset$. Let $H = M\backslash(X \cap \mathrm{cl}(A))$. Then $H|A = M|A$. As (M, A) is a minimal pair, it follows that H is not 3-connected. For some k in $\{1, 2\}$, let $\{Z, W\}$ be a k-separation of H for which $|Z \cap T_1| \geq 2$ and $|Z \cap F|$ is maximum subject to this condition.

5.15.1. *$F \not\subseteq Z$.*

Assume the contrary. Then $|Z| \geq 5$. But, by Lemma 5.14, $E(H) - E(M') \subseteq \{a_0, a_2, e_1, e_2\} \subseteq F$. Thus $Z \cap E(M') \supseteq Z - \{a_0, a_2, e_1, e_2\} \supseteq \{a_1\}$. Since $\{Z \cap$

$E(M'), W\}$ is not a 2-separation of M', it follows that $|Z \cap E(M')| \leq 1$. Hence $Z - \{a_0, a_2, e_1, e_2\} = \{a_1\}$, so $Z = F$.

Now a_0, a_1, and a_2 are in series on $M\backslash\{e_1, e_2\}$ because F is a fan. These elements are also in series in $M|A$. Thus they are also in series in $M|\text{cl}(A)$. Hence $\text{cl}(A)$ is spanned by $A - a_0$, so $\text{cl}(A) - a_0$ is spanned by $A - a_0$. But a_1 and a_2 are coloops of $M|(\text{cl}(A) - a_0)$. Thus $A - R$ spans $\text{cl}(A) - R$. Since $F = Z$, it follows that $A - R \subseteq W$. Hence W spans $\text{cl}(A) - R$. But $\text{cl}(A) - R \supseteq \text{cl}(A) \cap X$, so W spans $\text{cl}(A) \cap X$ and hence $\{Z, W \cup (X \cap \text{cl}(A))\}$ is a k-separation of M, a contradiction. Thus (5.15.1) holds.

5.15.2. $|W| = 2$ *and* W *is a cocircuit of* H.

Suppose that $|W| \geq 3$. Then a minor modification of the argument given in the last paragraph of the proof of Lemma 5.12 establishes that $F \subseteq Z$. This contradiction to (5.15.1) implies that $|W| \leq 2$. Then W consists of either one or two coloops of H, or a 2-cocircuit of H.

Suppose that W is a set of coloops of H. Then $W \subseteq \text{cl}(A)$ since every element in $E(M) - \text{cl}(A)$ belongs to a triangle of H. Orthogonality now implies that $W \subseteq A$. As W is a set of coloops of H, it is a set of coloops of $M|A$, so $W \cap R = \emptyset$. Hence W is a set of coloops of M'. This contradiction completes the proof of (5.15.2).

5.15.3. $W = \{a_0, a_2\}$.

Since W is a cocircuit of $M\backslash(X \cap \text{cl}(A))$, it follows that M has a cocircuit D that contains W and is contained in $W \cup (X \cap \text{cl}(A))$. Hence $D \subseteq W \cup (\text{cl}(A) - A)$. As $|D| \geq 3$, there is certainly an element of $\text{cl}(A) - A$ in D.

Now W does not contain a cocircuit of M', so $W \cap (Y \cup \{a_0, a_2\}) \neq \emptyset$. Suppose that $Y \cap W \neq \emptyset$. Then, by Lemma 5.14, we may assume that $\{e_1\} \subseteq Y \subseteq \{e_1, e_2\}$. But $Y \neq \{e_1, e_2\}$ otherwise a_1 is a loop of M/Y and hence of M'. Thus $Y = \{e_1\}$. Now, by orthogonality, since D meets $\text{cl}(A) - A$, it must also meet A. But the only element of D that can be in A is the unique element of of $W - Y$. Then, by orthogonality with T_1, this element must be a_1. Thus $|W \cap T_1| = 2$; a contradiction. We conclude that $Y \cap W = \emptyset$.

We may now assume that $a_0 \in W$. As $D \cap A \subseteq D \cap W$ and $D \cap A$ contains a cocircuit of $M|A$, we deduce, since R is a series class of $M|A$, that $D \cap W = \{a_0, a_2\}$ and so (5.15.3) holds.

5.15.4. $\{e_1, e_2\}$ *is a cocircuit of* H.

Since F is a fan of the 3-connected matroid M, the set $\{a_0, e_1, e_2, a_2\}$ is a cocircuit of M. Therefore this set is a union of cocircuits of H. By (5.15.2) and (5.15.3), $\{a_0, a_2\}$ is a cocircuit of H. Moreover, $\{e_1, a_1, e_2\}$ is a triangle of H. Assume that $\{e_1, e_2\}$ is not a cocircuit of H. Then, by orthogonality, $\{a_0, e_1, e_2\}$ or $\{a_2, e_1, e_2\}$ is a cocircuit of H. Thus $H|A$, which equals $M|A$, has $\{a_0\}$ or $\{a_2\}$ as a cocircuit. This contradiction finishes the proof of (5.15.4).

By (5.15.4), $\{\{e_1, e_2\}, E(H) - \{e_1, e_2\}\}$ is a 2-separation of H and $|\{e_1, e_2\} \cap T_1| \geq 2$. Thus the triangle T_1 implies that $\{\{e_1, e_2, a_1\}, E(H) - \{e_1, e_2, a_1\}\}$ is a 2-separation of H. Moreover, for each i in $\{0, 2\}$, the triad T_i of M implies that $\{\{e_1, e_2, a_1, a_i\}, E(H) - \{e_1, e_2, a_1, a_i\}\}$ is also a 2-separation of H unless $E(H) = F$. Assume that $E(H) \neq F$. Then H has a 2-separation $\{Z, W\}$ such that $Z \supseteq \{e_1, e_2, a_1, a_0\}$ and $|Z \cap F|$ is maximum subject to this. Then Lemma 5.14 implies that $W = \{a_0, a_2\}$; a contradiction. We conclude that $E(H) = F$. Then, since

T_1 is a triangle of H, and $\{a_0, a_2\}$ and $\{e_1, e_2\}$ are cocircuits of H, we deduce that $\{a_0, a_1, a_2\}$ is a triangle of H; a contradiction. We conclude that Lemma 5.15 holds. □

Now, for the minimal pair (M', A'), since $X \cup Y \subseteq \{e_1, e_2\}$, we have $\delta_E \leq 4$. Clearly $\delta_A = 2$. As $M'|A' = (M/\{a_0, a_2\})|A'$ and $\{a_0, a_1, a_2\}$ is a series class of $M|A$, we can obtain $M|A$ by taking the 2-sum of $M|A'$ and a 4-circuit. Thus $\delta_1 = 0$, $\delta_2 = 2$, $\delta_3 = 0$, and $\delta_\alpha = 0$. Finally, it is clear that $\delta_r \geq 0$. Thus, by Lemma 5.4, $\delta_m \geq 0$. Hence, by (10),

$$2 + 0 + 2 + 0 \leq \delta_A + \delta_1 + \delta_2 + \delta_m < \delta_\alpha + \delta_E \leq 0 + 4.$$

This contradiction completes the proof of Proposition 5.2. □

Theorem 5.1 is quite straightforward to prove by combining Proposition 5.2 and the following technical proposition that was proved in [6].

5.16. Proposition. *Let (M, A) be a minimal pair such that*

(i) *M is not isomorphic to $U_{1,3}$; and*
(ii) *every element of $E(M) - \mathrm{cl}(A)$ belongs to some type-2 fan of length three in which the rim is contained in a 4-circuit of $M|A$ and the spokes are contained in $E(M) - \mathrm{cl}(A)$.*

Then

$$|E(M)| \leq |A| + \lambda_1(A, M) + \lambda_2(A, M) - \beta(A, M),$$

where

$$\beta(A, M) = \begin{cases} 1, & \text{when } A \text{ is a circuit of } M \text{ or } r(A) \neq r(M); \\ 2, & \text{when } A \text{ is not a circuit of } M \text{ and } r(A) = r(M). \end{cases}$$

Proof of Theorem 5.1. Assume that the theorem fails and choose a minimal pair (M, A) that is a counterexample and is chosen so that $(|E(M)|, -|A|)$ is lexicographically minimal. Suppose first that $M \cong U_{1,3}$. Then $|A| = 3$ and it is straightforward to check that (M, A) is not a counterexample to the theorem. Thus we may assume that $M \not\cong U_{1,3}$. Moreover, by Proposition 5.2, (M, A) satisfies hypothesis (ii) of Proposition 5.16. Thus the latter proposition implies that

$$|E(M)| \leq |A| + \lambda_1(A, M) + \lambda_2(A, M) - \beta(A, M).$$

As $-\alpha(A, M) \geq -\beta(A, M)$ and $\min\{\lambda_3(A, M), r(M) - r_M(A)\} \geq 0$, it follows that

$$|E(M)| \leq |A| + \lambda_1(A, M) + \lambda_2(A, M) + \min\{\lambda_3(A, M), r(M) - r_M(A)\} - \alpha(A, M).$$

But this is a contradiction since (M, A) was chosen to be a counterexample to Theorem 5.1. □

To conclude the paper, we now prove Theorem 1.4.

Proof of Theorem 1.4. We shall prove that A is spanning. The theorem will then follow from [6, Theorem 1.1]. Suppose that $E(M) - \mathrm{cl}(A)$ is non-empty. Then it contains a cocircuit D of M. By Theorem 2.7, D must meet a triangle T of M. If $T \cap \mathrm{cl}(A)$ is empty, then, by Lemma 4.1(viii), M has a triad that contains two elements of T and one of A, so $M|A$ has a coloop; a contradiction. Thus T meets $\mathrm{cl}(A)$ so, by Lemma 4.1(ix), there is a chain T^*_{1T}, T, T^*_{2T} whose rim is contained in $\mathrm{cl}(A)$. Since $M|A$ has no coloops, this rim is contained in a series class S of $M|A$. Since $r(S) \leq 2$, it follows that S is a triangle of M. Thus if $X = T^*_{1T} \cup T \cup T^*_{2T}$, then

$$r(X) + r^*(X) - |X| \leq 3 + 3 - 5 = 1.$$

But M is 3-connected so $|E(M) - X| \leq 1$ and it follows without difficulty that M is a rank-3 wheel having $\text{cl}(A)$ as a triangle. Thus (M, A) is not a minimal pair; a contradiction. $\square$

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