ON K^4-FREE SUBGRAPHS OF RANDOM GRAPHS

Y. Kohayakawa, T. Łuczak, and V. Rödl

Abstract. For $0 < \gamma \leq 1$ and graphs G and H, write $G \to_\gamma H$ if any γ-proportion of the edges of G span at least one copy of H in G. As customary, write K^r for the complete graph on r vertices. We show that for every fixed real $\eta > 0$ there exists a constant $C = C(\eta)$ such that almost every random graph $G_{n,p}$ with $p = p(n) \geq Cn^{-2/5}$ satisfies $G_{n,p} \to_{2/3+\eta} K^4$. The proof makes use of a variant of Szemerédi's regularity lemma for sparse graphs and is based on a certain superexponential estimate for the number of pseudo-random tripartite graphs whose triangles are not too well distributed. Related results and a general conjecture concerning H-free subgraphs of random graphs in the spirit of the Erdős–Stone–Simonovits theorem are discussed.

§0. Introduction

A classical area of extremal graph theory investigates numerical and structural problems concerning *H-free graphs*, namely graphs that do not contain a copy of a given fixed graph H as a subgraph. Let $\mathrm{ex}(n, H)$ be the maximal number of edges that an H-free graph on n vertices may have. A basic question is then to determine or estimate $\mathrm{ex}(n, H)$ for any given H and large n. A solution to this problem is given by the celebrated Erdős–Stone–Simonovits theorem, which states that, as $n \to \infty$, we have

$$\mathrm{ex}(n, H) = \left(1 - \frac{1}{\chi(H) - 1} + o(1)\right)\binom{n}{2}, \tag{1}$$

where as usual $\chi(H)$ is the chromatic number of H. Furthermore, as proved independently by Erdős and Simonovits, every H-free graph $G = G^n$ that has as many edges as in (1) is in fact 'very close' (in a certain precise sense) to the densest n-vertex $(\chi(H) - 1)$-partite graph. For these and related results, see, for instance, Bollobás [**2**].

Here we are interested in a variant of the function $\mathrm{ex}(n, H)$. Let G and H be graphs, and write $\mathrm{ex}(G, H)$ for the maximal number of edges that an H-free subgraph of G may have. Formally, $\mathrm{ex}(G, H) = \max\{e(J) : H \not\subset J \subset G\}$, where $e(J)$ stands for the *size* $|E(J)|$ of J. Clearly $\mathrm{ex}(n, H) = \mathrm{ex}(K^n, H)$. As an example of

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a problem involving $\mathrm{ex}(G,H)$ with $G \neq K^n$, let us recall that a well-known conjecture of Erdős states that $\mathrm{ex}(Q^n, C^4) = (1/2 + o(1))e(Q^n)$, where Q^n stands for the n-dimensional hypercube and C^4 is the 4-cycle. (For several results concerning this conjecture, see Chung **[4]**.)

Our aim here is to study $\mathrm{ex}(G,H)$ when G is a 'typical' graph, by which we mean a *random graph*. Let $0 < p = p(n) \le 1$ and $0 < M = M(n) \le N = \binom{n}{2}$ be given. The standard binomial random graph $G_p = G_{n,p}$ has as vertex set a fixed set $V(G_p)$ of cardinality n and two such vertices are adjacent in G_p with probability p, with all such adjacencies independent. The random graph $G_M = G_{n,M}$ is simply a graph on a fixed n-element vertex set $V(G_M)$ chosen uniformly at random from all the $\binom{N}{M}$ possible candidates. (For concepts and results concerning random graphs not given in detail below, see *e.g.* Bollobás **[3]**.) Here we wish to investigate the random variables $\mathrm{ex}(G_{n,p}, H)$ and $\mathrm{ex}(G_{n,M}, H)$.

Let H be a graph of order $|H| = |V(H)| \ge 3$. Let us write $d_2(H)$ for the 2-*density* of H, that is

$$d_2(H) = \max\left\{\frac{e(J)-1}{|J|-2}\colon J \subset H,\ |J| \ge 3\right\}.$$

Given a real $0 \le \varepsilon \le 1$ and an integer $r \ge 2$, let us say that a graph J is ε-*quasi* r-*partite* if J may be made r-partite by the deletion of at most $\varepsilon e(J)$ of its edges. A general conjecture concerning $\mathrm{ex}(G_{n,p}, H)$ is as follows. For simplicity, below we restrict our attention to the binomial random graph $G_{n,p}$. Much of what follows may be restated in terms of $G_{n,M}$. As is usual in the theory of random graphs, we say that a property P holds *almost surely* or that *almost every* random graph $G_{n,p}$ or $G_{n,M}$ satisfies P if P holds with probability tending to 1 as $n \to \infty$.

Conjecture 1. *Let H be a non-empty graph of order at least 3, and let $0 < p = p(n) \le 1$ be such that $pn^{1/d_2(H)} \to \infty$ as $n \to \infty$. Then the following assertions hold.*

(i) *Almost every $G_{n,p}$ satisfies*

$$\mathrm{ex}(G_{n,p}, H) = \left(1 - \frac{1}{\chi(H)-1} + o(1)\right) e(G_{n,p}). \tag{2}$$

(ii) *Suppose $\chi(H) \ge 3$. Then for any $\varepsilon > 0$ there is a $\delta = \delta(\varepsilon) > 0$ such that almost every $G_{n,p}$ has the property that any H-free subgraph $J \subset G_{n,p}$ of $G_{n,p}$ with $e(J) \ge (1-\delta)\,\mathrm{ex}(G_{n,p}, H)$ is ε-quasi $(\chi(H)-1)$-partite.*

Recall that any graph G contains an r-partite subgraph $J \subset G$ with $e(J) \ge (1 - 1/r)e(G)$. Thus the content of Conjecture 1(i) is that $\mathrm{ex}(G_{n,p}, H)$ is at most as large as the right-hand side of (2), or, in other words, that $G_{n,p} \to_\gamma H$ holds almost surely for any fixed $\gamma > 1 - 1/(\chi(H)-1)$. There are a few results in support of Conjecture 1(i).

Any result concerning the tree-universality of expanding graphs or else a simple application of Szemerédi's regularity lemma for sparse graphs (see Lemma 4 below) give Conjecture 1(i) for forests. The cases in which $H = K^3$ and $H = C^4$ are essentially proved in Frankl and Rödl **[5]** and Füredi **[6]**, respectively, in connection with problems concerning the existence of some graphs with certain extremal properties. The case in which H is a general cycle was settled by Haxell, Kohayakawa, and Łuczak **[8, 9]** (see also Kohayakawa, Kreuter, and Steger **[11]**). Conjecture 1(ii)

in the case in which $0 < p < 1$ is a constant follows easily from Szemerédi's regularity lemma **[15]**. A variant of this lemma for sparse graphs (cf. Lemma 4 below) and a lemma from Kohayakawa, Łuczak, and Rödl **[12]** concerning induced subgraphs of bipartite graphs may be used to verify Conjecture 1 for $H = K^3$ in full. (See comments following Conjecture 23 for further details.) Still in the case in which $H = K^3$, for $0 < p < 1$ sufficiently close to $1/2$, a much stronger result than Conjecture 1(ii) was proved by Babai, Simonovits, and Spencer **[1]**. Finally, let us note that a result concerning *Ramsey* properties of random graphs in the spirit of Conjecture 1 was proved by Rödl and Ruciński **[13, 14]**.

Here we prove Conjecture 1(i) for $H = K^4$. Our results are as follows.

Theorem 2. *For any constant $0 < \eta \le 1/3$, there is a constant $C = C(\eta)$ for which the following holds. If $0 \le p = p(n) \le 1$ is such that $p \ge Cn^{-2/5}$ for all large enough n, then almost every $G_p = G_{n,p}$ is such that $G_p \to_{2/3+\eta} K^4$.*

Corollary 3. *For any constant $0 < \eta \le 1/3$, there is a constant $C = C(\eta)$ for which the following holds. If $0 \le M = M(n) \le \binom{n}{2}$ is such that $M \ge Cn^{8/5}$ for all large enough n, then almost every $G_M = G_{n,M}$ is such that $G_M \to_{2/3+\eta} K^4$.*

In §6 below, we formulate an auxiliary conjecture (Conjecture 23) that, if proved, would imply Conjecture 1 in full for all graphs H.

Finally, let us mention that Conjecture 1, if true, would immediately imply the existence of very 'sparse' graphs G satisfying the property that $G \to_\gamma H$ for any $\gamma > 1 - 1/(\chi(H) - 1)$. A simple corollary of Theorem 2 is that, for any $\eta > 0$, there is a graph $G = G_\eta$ that contains no K^5 but we have $G \to_{2/3+\eta} K^4$ (see §5). Erdős and Nešetřil have asked whether such graphs exist.

This note is organised as follows. In Section 1.1 we give a short outline of the proof of our main result, Theorem 2, and in Section 1.2, some preliminary results are given. In §2 the distribution of triangles in random and pseudo-random graphs is studied. In §3 we prove a key lemma in the proof of our main result, Lemma 16. Theorem 2 is proved in §4. In §5 we discuss a deterministic corollary to Theorem 2 concerning the Erdős–Nešetřil problem. Our last paragraph contains Conjecture 23.

§1. Outline of Proof and Preliminaries

1.1. Outline of the proof of Theorem 2. The proof of our main result is somewhat long and hence, for convenience, in this section we describe its main steps. Here we try to avoid being too technical.

The proof of Theorem 2 naturally splits into two parts. Suppose $p = p(n) \ge Cn^{-2/5}$, where C is some large constant, and let H be a spanning subgraph of $G_p = G_{n,p}$ with 'relative density' $e(H)/e(G_p) \ge \lambda$. Let us say that two vertices x, $y \in G_p$ are *K^4_--connected* by H if there are two other vertices z_1, $z_2 \in H$ such that both $\{x, z_1, z_2\}$ and $\{z_1, z_2, y\}$ induce triangles in H. Sometimes we also say that such a pair xy is a *pivotal pair*.

In the first part of our proof, we show that the number of pairs of vertices x, $y \in G_p$ that are K^4_--connected by H is roughly at least $(2\lambda - 1)\binom{n}{2}$, as long as C is a large enough constant. The precise statement of this result is given in Section 3.2, Lemma 16. The second part of the proof consists of deducing our Theorem 2 from Lemma 16. This part is less technical than the first, and is also considerably shorter. The method used here was inspired by an argument in Rödl and Ruciński **[14]**, and a version of this technique was used in Haxell, Kohayakawa, and Łuczak [**8**]. Let us give a brief description of this method.

Thus let $G_p = G_{n,p}$ be the binomial random graph with $p = p(n) \geq Cn^{-2/5}$, where C is some large constant, and let a constant $0 < \eta \leq 1/3$ be fixed. For simplicity, let us also assume that $p = p(n) \to 0$ as $n \to \infty$. We may write G_p as the union of k independent random graphs $G_{p_1}^{(j)}$ $(1 \leq j \leq k)$, where k is some large constant to be carefully chosen later. Since $p = o(1)$, below we may ignore the edges of $G_p = G_{p_1}^{(1)} \cup \dots \cup G_{p_1}^{(k)}$ that belong to more than one of the $G_{p_1}^{(j)}$. Let us now ask an 'adversary' to choose a subgraph $H \subset G_p$ of G_p of size at least $\lambda e(G_p)$, where $\lambda = 2/3 + \eta$, or, equivalently, let us ask our adversary to choose a set of edges $F \subset E(G_p)$ with $|F| \geq \lambda e(G_p)$. Our aim is to show that such a set F must span a K^4.

Instead of asking our adversary to pick F directly, we ask him to pick $F \cap E(G_{p_1}^{(j)})$ for all j. For some j_0, we must have $|F \cap E(G_{p_1}^{(j_0)}))| \geq \lambda e(G_{p_1}^{(j_0)})$. We may in fact ask our adversary to pick first j_0 and $F_{j_0} = F \cap E(G_{p_1}^{(j_0)})$, and leave the choice of $F \cap E(G_{p_1}^{(j)})$ $(j \neq j_0)$ for later. By Lemma 16, we know that at least $\sim (2\lambda - 1)\binom{n}{2} = (1/3 + 2\eta)\binom{n}{2}$ edges of K^n join pairs of vertices that are K_-^4-connected by F_{j_0}. We now show $G' = \bigcup_{j \neq j_0} G_{p_1}^{(j)}$ to our adversary, and ask him to pick $F \cap E(G')$. Note that, with very high probability, at least $1/3 + \eta$ of the edges of G' will be formed by K_-^4-connected pairs, and if our adversary puts any of these edges into $F \cap E(G')$, then F will span a copy of K^4. However, since G' contains an extremely large proportion of the edges of G_p (we choose k very large), our adversary is forced to pick at least $2/3$ of the edges of G', and hence he is forced to 'close' a K^4 by picking a K_-^4-connected pair xy for an edge of H.

Let us close this section with a few words on the proof of Lemma 16. Recall that in that lemma we are concerned with estimating the number of K_-^4-connected pairs induced by subgraphs of random graphs. A very simple lower bound for the number of such pairs induced by an arbitrary graph H_* is given in assertion (*) in Section 3.1. This estimate is far too weak to be of any use when dealing with subgraphs of random graphs, but a weighted version of this estimate, Lemma 15, is important in the proof of Lemma 16. Another important and a much deeper ingredient in the proof of Lemma 16 is a version of Szemerédi's regularity lemma [**15**] for sparse graphs; see Lemma 4 in Section 1.2 below. A simple application of Lemmas 4 and 15 allows us to focus our attention on certain ε-regular quadruples. The key lemma concerning such quadruples is Lemma 17 in Section 3.2. The proof of Lemma 17 is based on certain results concerning the number and the distribution of triangles in random and pseudo-random graphs. Paragraph 2 is entirely devoted to those results. The main lemmas in §2 are Lemmas 7 and 10.

1.2. Preliminaries. Let a graph $H = H^n$ of order $|H| = n$ be fixed. For U, $W \subset V = V(H)$ with $U \cap W = \emptyset$, we write $E(U,W) = E_H(U,W)$ for the set of edges of H that have one endvertex in U and the other in W. We set $e(U,W) = e_H(U,W) = |E(U,W)|$.

The following notion will be needed in what follows. Suppose $0 < \eta \leq 1$ and $0 < p \leq 1$. We say that H is η*-upper-uniform* with *density* p if, for all U, $W \subset V$ with $U \cap W = \emptyset$ and $|U|$, $|W| \geq \eta n$, we have $e_H(U,W) \leq (1+\eta)p|U||W|$. Clearly, if H is η-upper-uniform with density p, then it is also η'-upper-uniform with density p' for any $\eta \leq \eta' \leq 1$ and any $p \leq p' \leq 1$. In the sequel, for any two disjoint non-empty sets U, $W \subset V$, let

$$d_{H,p}(U,W) = e_H(U,W)/p|U||W|$$

be the *p-relative density* or, for short, the *p-density* of H between U and W. Now suppose $\varepsilon > 0$, U, $W \subset V$, and $U \cap W = \emptyset$. We say that the pair (U,W) is (ε,H,p)-*regular* if for all $U' \subset U$, $W' \subset W$ with $|U'| \geq \varepsilon|U|$ and $|W'| \geq \varepsilon|W|$ we have

$$|d_{H,p}(U',W') - d_{H,p}(U,W)| \leq \varepsilon.$$

We say that a partition $P = (V_i)_0^k$ of $V = V(H)$ is (ε,k)-*equitable* if $|V_0| \leq \varepsilon n$, and $|V_1| = \ldots = |V_k|$. Also, we say that V_0 is the *exceptional* class of P. When the value of ε is not relevant, we refer to an (ε,k)-equitable partition as a k-*equitable* partition. Similarly, P is an *equitable* partition of V if it is a k-equitable partition for some k. Finally, we say that an (ε,k)-equitable partition $P = (V_i)_0^k$ of V is (ε,H,p)-*regular* if at most $\varepsilon\binom{k}{2}$ pairs (V_i,V_j) with $1 \leq i < j \leq k$ are not (ε,p)-regular. We may now state an extension of Szemerédi's lemma **[15]** to subgraphs of η-upper-uniform graphs.

Lemma 4. *For any given $\varepsilon > 0$ and $k_0 \geq 1$, there are constants $\eta = \eta(\varepsilon,k_0) > 0$ and $K_0 = K_0(\varepsilon,k_0) \geq k_0$ that depend only on ε and k_0 such that any η-upper-uniform graph H with density $0 < p \leq 1$ admits an (ε,H,p)-regular (ε,k)-equitable partition of its vertex set with $k_0 \leq k \leq K_0$.* $\square$

Using standard estimates for tails of the binomial distribution, it is easy to check that a.e. $G_{n,p}$ is η-upper-uniform with density p for any constant $0 < \eta \leq 1$ if $d = pn$ is larger than some constant $d_0 = d_0(\eta)$.

Let us introduce a piece of notation before we proceed. If $U_1,\ldots,U_\ell \subset V(J)$ are pairwise disjoint sets of vertices of a given graph J, we write $J[U_1,\ldots,U_\ell]$ for the ℓ-partite subgraph of J naturally defined by the U_i ($1 \leq i \leq \ell$). Thus, $J[U_1,\ldots,U_\ell]$ has vertex set $\bigcup_1^\ell U_i$ and two of its vertices are adjacent if and only if they are adjacent in J and, moreover, they belong to distinct U_i.

Now suppose we have real numbers $0 < p \leq 1$, $0 < \varepsilon \leq 1$, $0 < \gamma_0 \leq 1$ and an integer $m \geq 1$. Suppose the above U_i ($1 \leq i \leq \ell$) all have cardinality m, and write γ_{ij} for the p-density $d_{J,p}(U_i,U_j)$ for all distinct i and j. Suppose L is a graph on $[\ell] = \{1,\ldots,\ell\}$ such that, for any $1 \leq i < j \leq \ell$, the pair (U_i,U_j) is (ε,J,p)-regular and $\gamma_{ij} \geq \gamma_0$ whenever $ij \in E(L)$.

We may now state our next lemma. In what follows, we write $O_1(x)$ for any term y satisfying $|y| \leq x$. Also, as usual, we write $\Delta = \Delta(L)$ for the maximal degree of L and we write $\Gamma_J(x)$ for the J-neighbourhood of a vertex $x \in V(J)$.

Lemma 5. *Let J, L, and the sets U_i ($1 \leq i \leq \ell$) be as above and let $\Delta = \Delta(L)$. Suppose $0 < \varepsilon \leq 1/(2\Delta+1)$ and put $\rho = (2\Delta + 1/\gamma_0)\varepsilon$ and $\mu = 2\Delta\varepsilon$. Then there are sets $\bar{U}_i \subset U_i$ with $|\bar{U}_i| \geq (1-\mu)m$ for all $1 \leq i \leq \ell$ such that, for all $x \in \bar{U}_i$ and any $1 \leq i \leq \ell$, we have*

$$d_{ij}(x) = |\Gamma_J(x) \cap \bar{U}_j| = (1 + O_1(\rho))\gamma_{ij}pm \tag{3}$$

for any j with $ij \in E(L)$. $\square$

Lemma 5 above is very similar to Lemma 2 in **[7]**, and hence its rather elementary proof is omitted. We close this section with a very simple large deviation inequality for the hypergeometric distribution. This inequality will be used in Section 2.1 below.

Lemma 6. *Let $1 \le a \le n$ and $1 \le t \le r \le n$ be integers, and suppose $R \subset [n]$ is an r-element subset of $[n] = \{1, \ldots, n\}$ chosen uniformly at random. Then*

$$\mathbb{P}\big(|R \cap [a]| \ge t\big) \le \left(\frac{a}{n-r+1}\right)^t \binom{r}{t}. \quad \square$$

§2. Triangles in Pseudo-Random and Random Graphs

2.1. The counting lemma. Let $m \ge 1$ be an integer. In this section we shall consider a fixed triple $\mathbf{V} = (\bar V_1, \bar V_2, \bar V_3)$ of pairwise disjoint sets with $m/2 \le m_i = |\bar V_i| \le m$ for all $i \in \{1,2,3\}$. We shall also suppose that $0 < p = p(m) \le 1$ satisfies $pm \ge m^{1/2+1/\log\log\log m}$ for all large enough m and, moreover, that $p = o(1)$ as $m \to \infty$. In this section, all the asymptotic notation refers to $m \to \infty$. Our aim is to estimate from above the number of certain pseudo-random tripartite graphs F with tripartition $V(F) = \bar V_1 \cup \bar V_2 \cup \bar V_3$ that contain unexpectedly few triangles given the number of edges that they have, or else whose triangles are not too regularly distributed.

Before we may describe precisely which graphs F are of interest to us, we need to introduce a few definitions. In what follows, indices will be tacitly taken modulo 3 when convenient. Let $0 < \delta \le 1$ be given. Suppose $e \in E_F(\bar V_{i-1}, \bar V_{i+1}) = E(F[\bar V_{i-1}, \bar V_{i+1}])$ $(i \in \{1,2,3\})$ and let $k_3(e) = k_3^F(e)$ be the number of triangles of F that contain e. We shall say that e is (δ, K^3)-*poor* if

$$k_3(e) < (1-\delta) d_{F,p}(\bar V_i, \bar V_{i-1}) d_{F,p}(\bar V_i, \bar V_{i+1}) p^2 m_i.$$

The graph F is (δ, K^3)-*unbalanced* if, for some $i \in \{1,2,3\}$, the number of (δ, K^3)-poor edges in $E_F(\bar V_{i-1}, \bar V_{i+1})$ is at least $\delta e_F(\bar V_{i-1}, \bar V_{i+1}) = \delta |E_F(\bar V_{i-1}, \bar V_{i+1})|$. For simplicity, below we write $\gamma_i = d_{F,p}(\bar V_{i-1}, \bar V_{i+1})$ $(i \in \{1,2,3\})$ and, if $x \in \bar V_i$ and $i \ne j \in \{1,2,3\}$, we let $d_{ij}(x) = |\Gamma_F(x) \cap \bar V_j|$.

Now suppose integers m and T and real constants $0 < \bar\varepsilon \le 1$, $0 < \bar\gamma_0 \le 1$, and $0 < \bar\rho \le 1$ are given, and let $\mathbf{V} = (\bar V_1, \bar V_2, \bar V_3)$ and $\mathbf{m} = (m_1, m_2, m_3)$ be as above. Let us write $\mathcal{F}_p(\bar\varepsilon, \bar\gamma_0, \bar\rho, T) = \mathcal{F}_p(\bar\varepsilon, \bar\gamma_0, \bar\rho; \mathbf{V}, T)$ for the set of tripartite graphs F with tripartition $V(F) = \bar V_1 \cup \bar V_2 \cup \bar V_3$ that satisfy the following properties:

(*i*) $(\bar V_1, \bar V_2, \bar V_3)$ is an $(\bar\varepsilon, F, p)$-regular triple,
(*ii*) $\bar\gamma_0 \le \gamma_i = d_{F,p}(\bar V_{i-1}, \bar V_{i+1}) \le 2$ for all $i \in \{1,2,3\}$,
(*iii*) $d_{ij}(x) = |\Gamma_F(x) \cap \bar V_j| = (1 + O_1(\bar\rho))\gamma_k p m_j$ for all $x \in \bar V_i$ and any choice of i, j, and k such that $\{i,j,k\} = [3]$,
(*iv*) F has size $e(F) = |E(F)| = T$.

Moreover, for any given $0 < \delta \le 1$, let $\mathcal{F}_p^\delta(\bar\varepsilon, \bar\gamma_0, \bar\rho, T) = \mathcal{F}_p^\delta(\bar\varepsilon, \bar\gamma_0, \bar\rho; \mathbf{V}, T)$ be the set of (δ, K^3)-unbalanced graphs F in $\mathcal{F}_p(\bar\varepsilon, \bar\gamma_0, \bar\rho, T)$. Put

$$f_p^\delta(\bar\varepsilon, \bar\gamma_0, \bar\rho; \mathbf{m}, T) = |\mathcal{F}_p^\delta(\bar\varepsilon, \bar\gamma_0, \bar\rho; \mathbf{V}, T)|.$$

Sometimes the labelling of the vertices of the graphs in $\mathcal{F}_p(\bar\varepsilon, \bar\gamma_0, \bar\rho; \mathbf{V}, T)$ or in $\mathcal{F}_p^\delta(\bar\varepsilon, \bar\gamma_0, \bar\rho; \mathbf{V}, T)$ is not relevant, and in that case we may replace $\mathbf{V}$ by $\mathbf{m}$ in our notation.

Our crucial counting lemma is as follows.

Lemma 7. *Let $0 < \alpha \le 1$, $0 < \bar\gamma_0 \le 1$, and $0 < \delta \le 1$ be given. Then there is a constant $\varepsilon_0 = \varepsilon_0(\alpha, \bar\gamma_0, \delta) > 0$ that depends only on α, $\bar\gamma_0$, and δ for which the following holds. Suppose $0 < p = p(m) \le 1$ is such that $p \ge m^{-1/2+1/\log\log\log m}$ for all large enough m and, moreover, $p = o(1)$ as $m \to \infty$. Then, if $\bar\rho \le \bar\rho_0 = \delta/27$, $\bar\varepsilon \le \varepsilon_0$, and m is sufficiently large, we have*

$$f_p^\delta(\bar\varepsilon, \bar\gamma_0, \bar\rho; \mathbf{m}, T) = |\mathcal{F}_p^\delta(\bar\varepsilon, \bar\gamma_0, \bar\rho; \mathbf{m}, T)| \le \alpha^T \binom{3m^2}{T}$$

for all T and all $\mathbf{m} = (m_1, m_2, m_3)$ with $m/2 \le m_i \le m$ ($i \in \{1, 2, 3\}$).

Most of the rest of Section 2.1 is dedicated to the proof of Lemma 7 above. Our general strategy in this proof is as follows. We randomly generate a tripartite graph F with tripartition $V(F) = \bar V_1 \cup \bar V_2 \cup \bar V_3$ and size $e(F) = |E(F)| = T$, and show that the probability that the graph we obtain will be a member of $\mathcal{F}_p^\delta(\bar\varepsilon, \bar\gamma_0, \bar\rho, T)$ is suitably small. We generate F in steps: we first generate $F[\bar V_2, \bar V_3]$. We then generate $F[\bar V_1, \bar V_3]$ and analyse the structure of the graph $F[\bar V_1, \bar V_3] \cup F[\bar V_2, \bar V_3]$. We then finally generate $F[\bar V_1, \bar V_2]$ and show that the appropriate probability is indeed small.

Let us now make precise the process by which we generate F. We first of all fix a partition $T = T_1 + T_2 + T_3$ of T such that, putting $\gamma_i = T_i/pm_{i-1}m_{i+1}$, we have $\bar\gamma_0 \le \gamma_i \le 2$ for all $i \in \{1, 2, 3\}$. Note that, because of condition (*ii*) above for F, we may disregard the T for which such a partition does not exist. Let us suppose that the bipartite graph $F_{23} = F[\bar V_2, \bar V_3]$ has been fixed, and that the following properties hold (cf. (*i*)–(*iv*) above): (*a*) the pair $(\bar V_2, \bar V_3)$ is $(\bar\varepsilon, F_{23}, p)$-regular, (*b*) $d_{ij}(x) = |\Gamma_{F_{23}}(x) \cap \bar V_j| = (1 + O_1(\bar\rho))\gamma_1 pm_j$ for all $x \in \bar V_i$, where $\{i, j\} = \{2, 3\}$, and (*c*) $e(F_{23}) = T_1$.

We now fix the degree sequence for the vertices $x \in \bar V_1$ in the bipartite graph $F[\bar V_1, \bar V_3]$, and generate this graph respecting this sequence. Thus let $(d_{13}(x))_{x\in\bar V_1}$ with $\sum_{x\in\bar V_1} d_{13}(x) = T_2$ and $d_{13}(x) = (1 + O_1(\bar\rho))\gamma_2 pm_3$ for all $x \in \bar V_1$ be fixed, and generate the bipartite graph $F_{13} = F[\bar V_1, \bar V_3]$ by selecting the neighbourhoods $\Gamma_{F_{13}}(x) \subset \bar V_3$ ($x \in \bar V_1$) randomly and independently for all $x \in \bar V_1$. Thus, for every $x \in \bar V_1$, all the $d_{13}(x)$-element subsets of $\bar V_3$ are equally likely to be chosen as the neighbourhood of x within $\bar V_3$. We now analyse the structure of $F_{13} \cup F_{23} = F[\bar V_1, \bar V_3] \cup F[\bar V_2, \bar V_3]$.

For convenience, let us put $d_{ij} = \mathrm{Ave}_{x\in\bar V_i} d_{ij}(x) = T_k/m_i = \gamma_k pm_j$ for all i, j, and k with $\{i, j, k\} = [3]$ and $k \ne 3$. Put also $d_{13} = \gamma_2 pm_3$ and $d_{23} = \gamma_1 pm_3$. Thus $d_{13}(x) = (1 + O_1(\bar\rho))d_{13}$ for all $x \in \bar V_1$, and $d_{23}(y) = (1 + O_1(\bar\rho))d_{23}$ for all $y \in \bar V_2$.

Let $0 < \beta_1 \le 1$ be given. For all $x \in \bar V_1$, put

$$\widetilde V_2 = \widetilde V_2(x, \beta_1) = \{y \in \bar V_2 \colon \sigma(x, y) \le -\beta_1 d_{13}d_{23}/m_3\},$$

where $\sigma(x, y) = d(x, y) - d_{13}d_{23}/m_3 = |\Gamma_{F_{13}}(x) \cap \Gamma_{F_{23}}(y)| - d_{13}d_{23}/m_3$. Note that the set $\widetilde V_2(x, \beta_1)$ is defined in such a way that the following fact holds: if $e = xy$ ($x \in \bar V_1$, $y \in \bar V_2$) is an edge of F, then e is a (β_1, K^3)-poor edge if and only if $y \in \widetilde V_2(x, \beta_1)$.

Now let $0 < \beta_2 \le 1$ be given. Below we say that $x \in \bar V_1$ is (β_1, β_2)-*faulty* if $|\widetilde V_2(x, \beta_1)| \ge \beta_2 m_2$. Note that, clearly, since we are conditioning on $F_{23} =$

$F[\bar{V}_2,\bar{V}_3]$, the event that a vertex $x\in\bar{V}_1$ should be (β_1,β_2)-faulty depends only on the random set $\Gamma_{F_{13}}(x)\subset\bar{V}_3$ that is chosen as the neighbourhood of x within $\bar{V}_3$.

Our next lemma is the key technical result in the proof of the main lemma in this section, Lemma 7.

Lemma 8. *Suppose the constants* $0<\beta_1\le 1$, $0<\beta_2\le 1$, $0<\bar{\gamma}_0\le 1$, $0<\bar{\varepsilon}\le 1$, *and* $0<\bar{\rho}\le 1$ *are such* $\beta_1\beta_2\ge 27\bar{\varepsilon}$, $\beta_2\bar{\rho}\le\bar{\varepsilon}$, *and* $\beta_2\le\bar{\gamma}_0$. *Then, for all sufficiently large* m, *the probability that a given vertex* $x\in\bar{V}_1$ *is* (β_1,β_2)*-faulty is at most* $\left(5\bar{\varepsilon}^{\bar{\varepsilon}/\beta_2}\right)^{d_{13}}$.

Proof. Let us fix $x\in\bar{V}_1$, and assume that $|\widetilde{V}_2|=|\widetilde{V}_2(x,\beta_1)|\ge\beta_2 m_2$. Let $\widetilde{V}_2^0\subset\widetilde{V}_2$ be such that $\widetilde{m}_2=|\widetilde{V}_2^0|=\lceil\beta_2 m_2\rceil$. The following assertion, whose proof we omit, is very similar to Lemma 5.

Assertion 1. There exist sets $\bar{V}_2'\subset\widetilde{V}_2^0$ *and* $\bar{V}_3'\subset\bar{V}_3$ *for which we have* $m_2'=|\bar{V}_2'|\ge(1-2\bar{\varepsilon}/\beta_2)\widetilde{m}_2$ *and* $m_3'=|\bar{V}_3'|\ge(1-2\bar{\varepsilon})m_3$, *and furthermore*

$$d_{23}'(y)=|\Gamma_{F_{23}}(y)\cap\bar{V}_3'|=\left(1+O_1\left(\frac{3\bar{\varepsilon}}{\beta_2}\right)\right)d_{23}\qquad(4)$$

for all $y\in\bar{V}_2'$, *and*

$$d_{32}'(z)=|\Gamma_{F_{23}}(z)\cap\bar{V}_2'|=\left(1+O_1\left(\frac{3\bar{\varepsilon}}{\beta_2}\right)\right)\beta_2 d_{32}\qquad(5)$$

for all $z\in\bar{V}_3'$.

Using Assertion 1, we prove next that there exists a *small* set Y of vertices from $\widetilde{V}_2$ whose F_{23}-neighbourhood uniformly cover essentially all of $\bar{V}_3$. We need to introduce some notation. Let us set $\omega=\omega(m)=m^{1/\log\log m}$, and let $d^Y(z)=|\Gamma_{F_{23}}(z)\cap Y|$ for all $Y\subset\bar{V}_2$ and all $z\in\bar{V}_3$. Put also

$$\bar{V}_3''=\bar{V}_3''(Y)=\left\{z\in\bar{V}_3\colon d^Y(z)=\left(1+O_1\left(\frac{4\bar{\varepsilon}}{\beta_2}\right)\right)\omega\right\}$$

for all $Y\subset\bar{V}_2$.

Assertion 2. There is a set $Y\subset\widetilde{V}_2$ *of cardinality* $q=|Y|=(1+O_1(3\bar{\varepsilon}/\beta_2))\omega m_2/d_{32}$ *such that* $|\bar{V}_3''|=|\bar{V}_3''(Y)|\ge(1-2\bar{\varepsilon})m_3$, *and such that*

$$d_{23}''(y)=|\Gamma_{F_{23}}(y)\cap\bar{V}_3''|\ge\left(1-\frac{3\bar{\varepsilon}}{\beta_2}\right)d_{23}\qquad(6)$$

for all $y\in Y$.

Let $\bar{V}_2'$ and $\bar{V}_3'$ be as in Assertion 1. To prove Assertion 2, we construct Y by randomly selecting its elements from the set $\bar{V}_2'\subset\widetilde{V}_2$. Let $p_Y=p_Y(m)=\omega/\beta_2 d_{32}$. Note that $0<p_Y<1$ for all large enough m. Put the vertices of $\bar{V}_2'$ into Y randomly, each with probability p_Y, and with all these events independent. The expected cardinality of Y is then

$$\mathbb{E}(|Y|)=\left(1+O_1\left(\frac{2\bar{\varepsilon}}{\beta_2}\right)\right)\frac{\omega m_2}{d_{32}},$$

and the expected degree of a vertex $z \in \bar{V}_3'$ into Y is

$$\mathbb{E}(d^Y(z)) = \left(1 + O_1\left(\frac{3\bar{\varepsilon}}{\beta_2}\right)\right)\omega.$$

From standard bounds for the tail of the binomial distribution, we may deduce that there is a set $Y \subset \bar{V}_2' \subset \widetilde{V}_2$ such that $q = |Y|$ is as required in Assertion 2, and such that every $z \in \bar{V}_3'$ satisfies $d^Y(z) = (1 + O_1(4\bar{\varepsilon}/\beta_2))\omega$. Note that, for such a set Y, we have $\bar{V}_3' \subset \bar{V}_3'' = \bar{V}_3''(Y)$, and hence $|\bar{V}_3''| \geq |\bar{V}_3'| \geq (1 - 2\bar{\varepsilon})m_3$. It now suffices to notice that every $y \in Y$ is such that

$$d_{23}''(y) = |\Gamma_{F_{23}}(y) \cap \bar{V}_3''| \geq |\Gamma_{F_{23}}(y) \cap \bar{V}_3'| \geq \left(1 - \frac{3\bar{\varepsilon}}{\beta_2}\right) d_{23},$$

since $y \in Y \subset \bar{V}_2'$ and relation (4) in Assertion 1 holds. This completes the proof of Assertion 2.

Assertion 3. The probability that our fixed vertex $x \in \bar{V}_1$ admits a set Y as in Assertion 2 is at most $\left(4\bar{\varepsilon}^{\bar{\varepsilon}/\beta_2}\right)^{d_{13}}$.

Let us first sketch the idea in the proof of Assertion 3. Roughly speaking, the set Y is such that the neighbourhoods $\Gamma_{F_{23}}(y) \cap \bar{V}_3''$ of the vertices $y \in Y$ within $\bar{V}_3''$ are about of the same size, and the vertices in $\bar{V}_3''$ are, by definition, covered by those sets quite uniformly. Now, since the vertices in Y are all in $\widetilde{V}_2$, we have that the neighbourhood $\Gamma_{F_{13}}(x)$ of x within $\bar{V}_3$ intersects all the neighbourhoods $\Gamma_{F_{23}}(y)$ $(y \in Y)$ too little. Thus it intersects the sets $\Gamma_{F_{23}}(y) \cap \bar{V}_3''$ too little as well, and this is possible only if it in fact intersects $\bar{V}_3''$ in an unexpectedly small set. It then suffices to estimate the probability that $\Gamma_{F_{13}}(x) \cap \bar{V}_3''$ should be as small. Let us now formalise the argument above.

Assume that a set Y as in Assertion 2 exists. We consider the vectors $\mathbf{g} = \mathbf{g}^x = (g_z)_{z \in \bar{V}_3''}$ and $\mathbf{f}^y = (f_z^y)_{z \in \bar{V}_3''}$ $(y \in Y)$ with entries

$$g_z = \begin{cases} 1 & \text{if } z \in \Gamma_{F_{13}}(x) \\ -d_{13}/m_3 & \text{otherwise,} \end{cases}$$

and

$$f_z^y = \begin{cases} 1 & \text{if } z \in \Gamma_{F_{23}}(y) \\ -d_{23}/m_3 & \text{otherwise.} \end{cases}$$

We first estimate $\xi_z = \sum_{y \in Y} f_z^y$ for $z \in \bar{V}_3''$. For any fixed $z \in \bar{V}_3''$, we have

$$\begin{aligned} \xi_z = \sum_{y \in Y} f_z^y = d^Y(z) - \frac{d_{23}}{m_3}\{q - d^Y(z)\} &= \left(1 + \frac{d_{23}}{m_3}\right) d^Y(z) - \frac{d_{23}q}{m_3} \\ &= \left(1 + \frac{d_{23}}{m_3}\right)\left(1 + O_1\left(\frac{4\bar{\varepsilon}}{\beta_2}\right)\right)\omega - \frac{d_{23}q}{m_3}. \end{aligned}$$

We have $d_{23} = \gamma_1 p m_3 = o(m_3)$, and, since $d_{23}/m_3 = T_1/m_2 m_3 = d_{32}/m_2$, we have $d_{23}q/m_3 = (1 + O_1(3\bar{\varepsilon}/\beta_2))\omega$. Therefore

$$\xi_z \leq (1 + o(1))\left(1 + \frac{4\bar{\varepsilon}}{\beta_2}\right)\omega - \left(1 - \frac{3\bar{\varepsilon}}{\beta_2}\right)\omega \leq \frac{8\bar{\varepsilon}}{\beta_2}\omega$$

for large enough m, and similarly $\xi_z \geq -(8\bar{\varepsilon}/\beta_2)\omega$ if m is sufficiently large. Given two vectors $\mathbf{a} = (a_z)_{z\in\bar{V}_3''}$ and $\mathbf{b} = (b_z)_{z\in\bar{V}_3''}$, let $\langle\mathbf{a},\mathbf{b}\rangle = \sum_{z\in\bar{V}_3''} a_z b_z$. We now estimate $\sum_{y\in Y}\langle\mathbf{f}^y,\mathbf{g}\rangle$. We have

$$\sum_{y\in Y}\langle\mathbf{f}^y,\mathbf{g}\rangle = \left\langle\sum_{y\in Y}\mathbf{f}^y,\mathbf{g}\right\rangle = \left\langle(\xi_z)_{z\in\bar{V}_3''},\mathbf{g}\right\rangle = \sum_{z\in\bar{V}_3''}\xi_z g_z.$$

Therefore

$$\left|\sum_{y\in Y}\langle\mathbf{f}^y,\mathbf{g}\rangle\right| \leq \sum_{z\in\bar{V}_3''}|\xi_z g_z| \leq \frac{8\bar{\varepsilon}}{\beta_2}\omega\sum_{z\in\bar{V}_3''}|g_z|.$$

We have from our assumptions on our constants that $\bar{\rho} \leq \bar{\varepsilon}/\beta_2 \leq \beta_1/27 \leq 1/27$. Hence

$$\sum_{z\in\bar{V}_3''}|g_z| \leq \sum_{z\in\bar{V}_3}|g_z| \leq (1+\bar{\rho})d_{13} + d_{13} \leq \frac{9}{4}d_{13},$$

with plenty to spare. Thus

$$\left|\sum_{y\in Y}\langle\mathbf{f}^y,\mathbf{g}\rangle\right| \leq \frac{18\bar{\varepsilon}}{\beta_2}\omega d_{13}. \tag{7}$$

We now give a lower estimate for the left-hand side of (7) in terms of the $\sigma(x,y)$ $(y\in Y)$. Let $d_{13}''(x) = |\Gamma_{F_{13}}(x)\cap\bar{V}_3''|$ and recall that we let $d_{23}''(y) = |\Gamma_{F_{23}}(y)\cap\bar{V}_3''|$ for all $y\in Y$. Put $m_3'' = |\bar{V}_3''|$ and $d''(x,y) = |\Gamma_{F_{13}}(x)\cap\Gamma_{F_{23}}(y)\cap\bar{V}_3''|$ for all $y\in Y$. Fix $y\in Y$. Clearly $d''(x,y)\leq d(x,y) = |\Gamma_{F_{13}}(x)\cap\Gamma_{F_{23}}(y)|$. Thus

$$\begin{aligned}\langle\mathbf{f}^y,\mathbf{g}\rangle = d''(x,y) - \frac{d_{13}}{m_3}\big\{d_{23}''(y) - d''(x,y)\big\} - \frac{d_{23}}{m_3}\big\{d_{13}''(x) - d''(x,y)\big\}\\ + \frac{d_{13}d_{23}}{m_3^2}\Big(m_3'' - \big\{d_{13}''(x) + d_{23}''(y) - d''(x,y)\big\}\Big).\end{aligned}$$

Write $d_{13}''(x) = (1-\tau)d_{13}$. Recalling (6), we see that

$$\begin{aligned}\langle\mathbf{f}^y,\mathbf{g}\rangle &\leq d(x,y) - 2\frac{d_{13}d_{23}}{m_3} + \frac{d_{13}d_{23}m_3''}{m_3^2}\\ &\quad + \frac{3\bar{\varepsilon}d_{13}d_{23}}{\beta_2 m_3} + \tau\frac{d_{13}d_{23}}{m_3} + \frac{d_{13}d''(x,y)}{m_3} + \frac{d_{23}d''(x,y)}{m_3}\\ &\quad - \frac{d_{13}d_{23}d_{13}''(x)}{m_3^2} - \frac{d_{13}d_{23}d_{23}''(y)}{m_3^2} + \frac{d_{13}d_{23}d''(x,y)}{m_3^2}\\ &\leq \sigma(x,y) + \frac{3\bar{\varepsilon}d_{13}d_{23}}{\beta_2 m_3} + \tau\frac{d_{13}d_{23}}{m_3} + \frac{d_{13}+d_{23}}{m_3}d''(x,y) + o\left(\frac{d_{13}d_{23}}{m_3}\right),\end{aligned}$$

where we used that, trivially, $m_3''\leq m_3$, and that $d_{13}(x)$, $d_{23}(y)$, and $d''(x,y)$ are all $o(m_3)$, since by assumption $p = p(m)\to 0$ as $m\to\infty$. However,

$$\begin{aligned}\frac{d_{13}+d_{23}}{m_3}\sum_{y\in Y}d''(x,y) &= \frac{d_{13}+d_{23}}{m_3}\sum_{y\in Y}|\Gamma_{F_{13}}(x)\cap\Gamma_{F_{23}}(y)\cap\bar{V}_3''|\\ &\leq \frac{d_{13}+d_{23}}{m_3}\sum\left\{d^Y(z)\colon z\in\Gamma_{F_{13}}(x)\cap\bar{V}_3''\right\}\\ &\leq \frac{d_{13}+d_{23}}{m_3}(1+\bar{\rho})\left(1+\frac{4\bar{\varepsilon}}{\beta_2}\right)\omega d_{13},\end{aligned}$$

which is $o(d_{13}d_{23}q/m_3)$. Therefore

$$\sum_{y\in Y}\langle \mathbf{f}^y, \mathbf{g}\rangle \le -\left(\beta_1 - \frac{4\bar\varepsilon}{\beta_2} - \tau\right)\frac{d_{13}d_{23}q}{m_3}. \tag{8}$$

Using that $q = (1 + O_1(3\bar\varepsilon/\beta_2))\omega m_2/d_{32}$ and that $d_{23}/m_3 = d_{32}/m_2$, we deduce from inequality (8) above that

$$\begin{aligned}\sum_{y\in Y}\langle \mathbf{f}^y, \mathbf{g}\rangle &\le -\left(\beta_1 - \frac{4\bar\varepsilon}{\beta_2} - \tau\right)\left(1 + O_1\left(\frac{3\bar\varepsilon}{\beta_2}\right)\right)\omega d_{13}\\ &\le -\left(\beta_1 - \frac{4\bar\varepsilon}{\beta_2} - \tau + O_1\left(\frac{3\bar\varepsilon}{\beta_2}\right)\right)\omega d_{13}. \end{aligned}\tag{9}$$

We now claim that $\tau \ge 2\bar\varepsilon/\beta_2$. Indeed, otherwise we would have that $\beta_1 - 4\bar\varepsilon/\beta_2 - \tau + O_1(3\bar\varepsilon/\beta_2) \ge 0$, and hence, comparing inequalities (7) and (9), we would have that $\beta_1 - 4\bar\varepsilon/\beta_2 - \tau + O_1(3\bar\varepsilon/\beta_2) \le 18\bar\varepsilon/\beta_2$, and consequently that $\tau \ge 2\bar\varepsilon/\beta_2$, which is a contradiction.

Recall that $d''_{13}(x) = |\Gamma_{F_{13}}(x) \cap \bar V''_3| = (1-\tau)d_{13}$ and that $d_{13}(x) = |\Gamma_{F_{13}}(x)| = (1 + O_1(\bar\rho))d_{13}$. Also, $|\bar V_3 \setminus \bar V''_3| \le 2\bar\varepsilon|\bar V_3|$. Thus we deduce from the existence of the set Y that at least $(\tau - \bar\rho)d_{13} \ge \tau d_{13}/2$ elements of $\Gamma_{F_{13}}(x) \subset \bar V_3$ are confined to a subset of $\bar V_3$ of cardinality at most $2\bar\varepsilon|\bar V_3|$. Thus, by Lemma 6, the probability that such a set Y exists is at most $2^{(1+\bar\rho)d_{13}}(3\bar\varepsilon)^{\tau d_{13}/2} \le 2^{3d_{13}/2}(3\bar\varepsilon)^{\bar\varepsilon d_{13}/\beta_2} \le \left(4\bar\varepsilon^{\bar\varepsilon/\beta_2}\right)^{d_{13}}$, completing the proof of Assertion 3.

We may now finish the proof of Lemma 8 combining Assertions 2 and 3 above. Indeed, writing $\sum'_q$ for the sum over all q satisfying $q = (1 + O_1(3\bar\varepsilon/\beta_2))\omega m_2/d_{32}$, we have from the above two assertions that probability that the vertex x should be (β_1, β_2)-faulty is, for large enough m, at most

$$\begin{aligned}\left(4\bar\varepsilon^{\bar\varepsilon/\beta_2}\right)^{d_{13}}{\sum}'_q\binom{m_2}{q} &\le \frac{6\bar\varepsilon\omega m_2}{\beta_2 d_{32}}\binom{m_2}{2\omega m_2/d_{32}}\left(4\bar\varepsilon^{\bar\varepsilon/\beta_2}\right)^{d_{13}}\\ &\le \frac{6\bar\varepsilon\omega m_2}{\beta_2 d_{32}}\left(\frac{\mathrm{e}d_{32}}{2\omega}\right)^{2\omega m_2/d_{32}}\left(4\bar\varepsilon^{\bar\varepsilon/\beta_2}\right)^{d_{13}} \le \left(5\bar\varepsilon^{\bar\varepsilon/\beta_2}\right)^{d_{13}},\end{aligned}$$

where in the last inequality we used that $\omega m_2(\log m)/d_{32} = o(d_{13})$, which follows easily from our assumption that $pm \ge m^{1/2+1/\log\log\log m}$ for all sufficiently large m. Thus Lemma 8 is proved. □

We are now in position to finish the proof of Lemma 7.

Proof of Lemma 7. Let us first state a simple fact concerning the T_i.

Assertion 1. For all $i \in \{1,2,3\}$ we have $T_i \ge (\bar\gamma_0/24)T$.

Indeed, $T \le \sum_{i\in\{1,2,3\}}\gamma_i pm_{i-1}m_{i+1} \le 6pm^2$, and therefore, for any $i \in \{1,2,3\}$, we have

$$T_i = \gamma_i pm_{i-1}m_{i+1} \ge \frac{\bar\gamma_0}{4}pm^2 \ge \frac{\bar\gamma_0}{24}T,$$

as required.

Our next observation is an immediate consequence of Lemma 8 and Assertion 1.

Assertion 2. Let $0 < \beta_3 \le 1$ be given. The probability that at least $\beta_3 m_1$ vertices in $\bar V_1$ are (β_1, β_2)-faulty is at most $\big(6\bar\varepsilon^{\bar\varepsilon/\beta_2}\big)^{\beta_3\bar\gamma_0 T/24}$.

Indeed, by Lemma 8 and Assertion 1 above, the probability in question is at most

$$2^{m_1}\big((5\bar\varepsilon^{\bar\varepsilon/\beta_2})^{d_{13}}\big)^{\beta_3 m_1} = 2^{m_1}\big(5\bar\varepsilon^{\bar\varepsilon/\beta_2}\big)^{\beta_3 T_1} \le \big(6\bar\varepsilon^{\bar\varepsilon/\beta_2}\big)^{\beta_3\bar\gamma_0 T/24},$$

completing the proof of Assertion 2.

We now describe the last step in the generation of F. Recall that we have generated $F_{13} \cup F_{23} = F[\bar V_1, \bar V_3] \cup F[\bar V_2, \bar V_3]$ so far. Let $K[\bar V_1, \bar V_2]$ be the complete bipartite graph with bipartition $\bar V_1 \cup \bar V_2$. To generate $F_{12} = F[\bar V_1, \bar V_2]$, we randomly pick for $E(F[\bar V_1, \bar V_2])$ a T_3-element subset of $E(K[\bar V_1, \bar V_2])$, uniformly chosen from all such sets.

Assertion 3. Suppose that fewer than $\beta_3 m_1$ vertices in $\bar V_1$ are (β_1, β_2)-faulty. Then the probability that at least δT_3 edges in $F_{12} = F[\bar V_1, \bar V_2]$ are (β_1, K^3)-poor is no larger than $\big\{4(\beta_2+\beta_3)^{\delta\bar\gamma_0/24}\big\}^T$.

Recall that an edge $xy \in E(F[\bar V_1, \bar V_2])$ ($x \in \bar V_1$, $y \in \bar V_2$) is (β_1, K^3)-poor if and only if $y \in \widetilde V_2(x, \beta_1)$. The probability in question P_0 is the probability that at least δT_3 edges $xy \in E(K[\bar V_1, \bar V_2])$ ($x \in \bar V_1$, $y \in \bar V_2$) with $y \in \widetilde V_2(x, \beta_1)$ are selected to be elements of F_{12}. The number of such 'potentially poor' edges xy in $K[\bar V_1, \bar V_2]$ is at most $\beta_3 m_1 m_2 + (1-\beta_3)\beta_2 m_1 m_2 \le (\beta_2+\beta_3) m_1 m_2$, and hence, by Lemma 6 and Assertion 1, we have

$$P_0 \le 2^{T_3}\big\{2(\beta_2+\beta_3)\big\}^{\delta T_3} \le \big\{4(\beta_2+\beta_3)^{\delta}\big\}^{\bar\gamma_0 T/24} \le \big\{4(\beta_2+\beta_3)^{\delta\bar\gamma_0/24}\big\}^T,$$

proving Assertion 3.

We may now finish the proof of Lemma 7. Let the constants α, $\bar\gamma_0$, and δ as in the statement of our lemma be given. We then let $\varepsilon_0 = \varepsilon_0(\alpha, \bar\gamma_0, \delta)$ be such that $0 < \varepsilon_0 \le \delta\bar\gamma_0/27$ and, moreover,

$$\big(6\bar\varepsilon^{\delta/27}\big)^{\bar\gamma_0/24 \log\log(1/\bar\varepsilon)} \le \frac{1}{6}\alpha$$

and

$$4\left(\frac{27\bar\varepsilon}{\delta} + \frac{1}{\log\log(1/\bar\varepsilon)}\right)^{\delta\bar\gamma_0/24} \le \frac{1}{6}\alpha$$

for all $0 < \bar\varepsilon \le \varepsilon_0$.

We now apply Assertions 2 and 3 above with suitably chosen β_1, β_2, β_3, $\bar\varepsilon$, and $\bar\rho$. Let $\beta_1 = \delta$ and fix any $0 < \bar\varepsilon \le \varepsilon_0$. Let $\beta_2 = 27\bar\varepsilon/\delta \le \bar\gamma_0$ and $\beta_3 = 1/\log\log(1/\bar\varepsilon)$. Recall that $\bar\rho_0 = \delta/27$. To complete the proof, we proceed as follows: we suppose that $\bar\rho \le \bar\rho_0$ is given and that m is sufficiently large for the inequalities below to hold. Fix the partition $T = T_1 + T_2 + T_3$ of T, the bipartite graph $F_{23} = F[\bar V_2, \bar V_3]$, and the degree sequence $(d_{13}(x))_{x\in\bar V_1}$ as above. Then generate $F_{13} = F[\bar V_1, \bar V_3]$. The probability that at least $\beta_3 m_1$ vertices in $\bar V_1$ are (β_1, β_2)-faulty is, by Assertion 2, at most

$$\big(6\bar\varepsilon^{\bar\varepsilon/\beta_2}\big)^{\beta_3\bar\gamma_0 T/24} = \big(6\bar\varepsilon^{\delta/27}\big)^{\beta_3\bar\gamma_0 T/24} \le \left(\frac{\alpha}{6}\right)^T \le \frac{1}{6}\alpha^T.$$

Let us now assume that fewer than $\beta_3 m_1$ vertices in $\bar{V}_1$ are (β_1, β_2)-faulty, and let us generate $F_{12} = F[\bar{V}_1, \bar{V}_2]$. Then the probability that δT_3 edges in $F_{12} = F[\bar{V}_1, \bar{V}_2]$ are (δ, K^3)-poor is, by Assertion 3, at most

$$\left\{4(\beta_2+\beta_3)^{\delta\bar{\gamma}_0/24}\right\}^T \le \left\{4\left(\frac{27\bar{\varepsilon}}{\delta} + \frac{1}{\log\log(1/\bar{\varepsilon})}\right)^{\delta\bar{\gamma}_0/24}\right\}^T \le \left(\frac{\alpha}{6}\right)^T \le \frac{1}{6}\alpha^T.$$

We now note that the argument above is symmetric with respect to the indices $i \in \{1,2,3\}$ (note the factor '3' below), and thus we may conclude that

$$|\mathcal{F}_p^\delta(\bar{\varepsilon}, \bar{\gamma}_0, \bar{\rho}; \mathbf{V}, T)| \le 3 \times \frac{2}{6}\alpha^T \binom{3m^2}{T} \le \alpha^T \binom{3m^2}{T},$$

completing the proof of Lemma 7. $\square$

Recall that m and T are integers, $0 < \bar{\varepsilon} \le 1$, $0 < \bar{\gamma}_0 \le 1$, $0 < \delta \le 1$, and $0 < \bar{\rho} \le 1$ are fixed reals, $0 < p = p(m) \le 1$ is such that $pm \to \infty$ and $p = o(1)$ as $m \to \infty$, and $\mathbf{V} = (\bar{V}_1, \bar{V}_2, \bar{V}_3)$ and $\mathbf{m} = (m_1, m_2, m_3)$, where the $\bar{V}_i$ $(i \in \{1,2,3\})$ are pairwise disjoint sets with cardinality $|\bar{V}_i| = m_i$ $(i \in \{1,2,3\})$. Our next lemma concerns a property of graphs $F \in \mathcal{F}_p(\bar{\varepsilon}, \bar{\gamma}_0, \bar{\rho}, T) \setminus \mathcal{F}_p^\delta(\bar{\varepsilon}, \bar{\gamma}_0, \bar{\rho}, T)$, namely, graphs F in $\mathcal{F}_p(\bar{\varepsilon}, \bar{\gamma}_0, \bar{\rho}, T)$ that are (δ, K^3)-*balanced*. For a vertex $x \in V(F)$, let $k_3(x) = k_3^F(x)$ denote the number of triangles of F that contain x.

Lemma 9. *Suppose F is a (δ, K^3)-balanced graph in $\mathcal{F}_p(\bar{\varepsilon}, \bar{\gamma}_0, \bar{\rho}; \mathbf{V}, T)$. Put $\delta' = (2\delta)^{1/2}$. Then, for any $i \in \{1,2,3\}$, there are at most $\delta' m_i$ vertices x in $\bar{V}_i$ such that*

$$k_3(x) = k_3^F(x) < (1 - \bar{\rho} - \delta')\gamma_1\gamma_2\gamma_3 p^3 m_{i-1} m_{i+1}.$$

Proof. By symmetry, it suffices to prove the statement for $i = 1$. In the sequel we freely use the notation introduced before Lemma 7. Also, let us put $d_{ij} = \mathrm{Ave}_{x \in \bar{V}_i} d_{ij}(x) = \gamma_k p m_j$ for all i, j, and k with $\{i,j,k\} = [3]$. Below we say that a vertex $x \in \bar{V}_1$ is *bad* if at least $\delta d_{12}/\delta' = \delta' d_{12}/2$ edges in $E(F[\bar{V}_1, \bar{V}_2])$ incident to x are (δ, K^3)-poor edges. The number of such bad vertices $x \in \bar{V}_1$ is at most $\delta' m_1$, as otherwise the number of (δ, K^3)-poor edges in $E(F[\bar{V}_1, \bar{V}_2])$ would be more than $\delta m_1 d_{12} = \delta e(F[\bar{V}_1, \bar{V}_2])$, contradicting the fact that F is (δ, K^3)-balanced.

Note that an edge $e = x_1 x_2 \in E(F[\bar{V}_1, \bar{V}_2])$ $(x_i \in \bar{V}_i,\ i \in \{1,2\})$ that is not (δ, K^3)-poor 'contributes' with at least $k_3(e) \ge (1-\delta)\gamma_1\gamma_2 p^2 m_3$ triangles to $k_3(x_1)$. Supposing that $x_1 \in \bar{V}_1$ is not a bad vertex, summing over all $x_2 \in \bar{V}_2$ for which $x_1 x_2 \in E(F[\bar{V}_1, \bar{V}_2])$ is not a (δ, K^3)-poor edge, we obtain that

$$\begin{aligned} k_3(x_1) &\ge \left((1-\bar{\rho})d_{12} - \frac{\delta'}{2} d_{12}\right)(1-\delta)\gamma_1\gamma_2 p^2 m_3 \\ &\ge \left(1 - \bar{\rho} - \delta - \frac{\delta'}{2}\right)\gamma_1\gamma_2\gamma_3 p^3 m_2 m_3 \ge (1 - \bar{\rho} - \delta')\gamma_1\gamma_2\gamma_3 p^3 m_2 m_3. \end{aligned}$$

Since, as we saw above, at most $\delta' m_1$ vertices $x_1 \in \bar{V}_1$ are bad, the proof is complete. $\square$

2.2. Distribution of triangles in random graphs. In this section we look at the random graph $G_p = G_{n,p}$ and study the distribution of the triangles it contains. The aim will be to prove that the triangles of $G_{n,p}$ do not unduly concentrate on any fixed set of edges and vertices. To be precise, let x be any given vertex of G_p and let E be a set of *edges* of G_p taken from the subgraph $G_p[\Gamma_{G_p}(x)]$ induced by the neighbourhood $\Gamma_{G_p}(x)$ of x in G_p. Also, let W be a subset of *vertices* of G_p disjoint from $\{x\}\cup V(E)$, where we write $V(E)$ for the set $V(G_p[E])$ of vertices of G_p that are incident to at least one edge from E. Our aim is to find an upper bound for the number of triangles $k_3(E,W) = k_3^{G_p}(E,W)$ of G_p that are determined by an edge from E and a vertex in W. Note that the expected value of this number is $p^2|E||W|$. We shall show that this is an upper bound in probability up to $1+\theta$ for any fixed $\theta > 0$ as long as E and W are reasonably large and $p = p(n)$ does not tend to 0 too fast.

Lemma 10. *Let c_1, $c_2 > 0$ and $0 < \theta \le 1$ be given. Then there is a constant $C_0 = C_0(c_1, c_2, \theta)$ that depends only on c_1, c_2, and θ for which the following holds. Suppose $p = p(n) = \omega n^{-2/5}$ where $C_0 \le \omega = \omega(n) = o(n^{1/190})$. Then almost every $G_p = G_{n,p}$ is such that, if $E \subset E(G_p[\Gamma_{G_p}(x)])$ and $W \subset \widetilde{W} = V(G_p) \setminus (\{x\} \cup V(E))$ for some $x \in V(G_p)$, then*

$$k_3(E,W) = k_3^{G_p}(E,W) \le (1+\theta)p^2|E||W| \tag{10}$$

as long as $|E| \ge c_1 p^3 n^2$ and $|W| \ge c_2 n$.

Most of the remainder of this section is devoted to the proof of Lemma 10. Unfortunately, our proof below is a little indirect and is based on a few auxiliary lemmas; moreover, this proof makes use of a technical condition that ω should not be too large. The obvious direct approaches based on simple large deviation inequalities seem to fail to give Lemma 10. To see why this might be expected, note that (i) for the sets E and W of interest, the expected value $\mathbb{E}(k_3(E,W)) = p^2|E||W|$ of $k_3(E,W)$ is of order $O(n)$ only, while the number of sets W that we have to handle is $\exp\{\Omega(n)\}$, and (ii) $k_3(E,W)$ is a sum of *positively correlated* indicator variables and the most common large deviation inequalities for such sums do not seem to be strong enough for our purposes.

Let us turn to the proof of Lemma 10. For the rest of this section we assume that $p = p(n) = \omega n^{-2/5}$, where $C_0 \le \omega = \omega(n) = o(n^{1/190})$ and C_0 is some large constant. (The main result for G_p with larger p will be deduced from the small p case; cf. Lemma 19.)

Let P^3 be the path of length 2 and E^k the k-vertex graph with no edges ($k \ge 1$). We write H_k for $P^3 \vee E^k$ ($k \ge 1$), the graph on $k+3$ vertices we obtain from the disjoint union of P^3 and E^k by adding all the $3k$ edges between $V(P^3)$ and $V(E^k)$. A little piece of arithmetic shows that almost no $G_{n,p}$ contains a copy of H_{12}. Thus for the rest of this section we may and shall assume that our $G_{n,p}$ is H_{12}-free.

We may clearly assume that the degree of any vertex of $G_p = G_{n,p}$ is $(1+o(1))pn$, and also that any vertex of G_p is contained in at most p^3n^2 triangles. Furthermore, the expected number of common neighbours of any two fixed vertices of G_p is $p^2(n-2) \le \omega^2 n^{1/5}$. Thus, we may and shall condition on our G_p being such that $d_{G_p}(x,y) = |\Gamma_{G_p}(x) \cap \Gamma_{G_p}(y)| \le 2\omega^2 n^{1/5}$ for any pair of distinct vertices x, $y \in V(G_p)$. Finally, we may assume that G_p is $o(1)$-upper-uniform.

Suppose $x \in V(G_p)$ and $E \subset E(G_p[\Gamma_{G_p}(x)])$. For a vertex $y \in \widetilde{W} = V(G_p) \setminus (\{x\} \cup V(E))$, let E_y be the set $E \cap E(G_p[\Gamma_{G_p}(y)])$ of edges of E that the neighbourhood of y in G_p induces in G_p. Clearly, $k_3(E, y) = k_3(E, \{y\}) = |E_y|$. We shall say that a vertex $y \in \widetilde{W}$ is *(x, E)-bad*, or simply *E-bad*, if E_y is not an independent set of edges.

Lemma 11. *Almost every G_p is such that, for any $x \in V(G_p)$ and any $E \subset E(G_p[\Gamma_{G_p}(x)])$, at most $10e\omega^8 n^{4/5}$ vertices are E-bad.*

Proof. Fix a vertex $x \in V(G_p)$. Let us generate G_p as follows: we first choose the neighbourhood $\Gamma_{G_p}(x)$ of x in G_p, and once this set is fixed, we decide which edges within $\Gamma_{G_p}(x)$ should be in G_p. Put $E^{(0)} = E(G_p[\Gamma_{G_p}(x)])$, and let $V(E^{(0)}) \subset \Gamma_{G_p}(x)$ be the set of vertices in $\Gamma_{G_p}(x)$ that are incident to at least one edge in $E(G_p[\Gamma_{G_p}(x)])$. For the rest of the proof, we assume that the edges generated so far are fixed.

Let us now consider a vertex $y \in Y = V(G_p) \setminus (\{x\} \cup \Gamma_{G_p}(x))$, and let us decide which edges between y and $V(E^{(0)})$ should be in our G_p. Notice that whether or not y is $E^{(0)}$-bad depends solely on these y–$V(E^{(0)})$ edges. In particular, the events 'y is $E^{(0)}$-bad' ($y \in Y$) are all independent. Let us estimate the probability that a given vertex $y \in Y$ turns out to be $E^{(0)}$-bad. We first observe that with probability $1 - o(1/n)$ we have $|V(E^{(0)})| \le |\Gamma_{G_p}(x)| \le 2pn$ and $|\Delta(E^{(0)})| = |\Delta(G_p[E^{(0)}])| \le 2p^2 n = 2\omega^2 n^{1/5}$. In the sequel, we assume that these two inequalities hold. In particular, the number of copies of P^3 spanned by $E^{(0)}$ is, crudely, at most $|V(E^{(0)})|\Delta(E^{(0)})^2 \le 8\omega^5 n$. Thus the probability that y is $E^{(0)}$-bad is

$$\begin{aligned}\mathbb{P}(\Delta(E_y^{(0)}) \ge 2) &= \mathbb{P}(G_p[E_y^{(0)}] \text{ contains a } P^3) \\ &\le \mathbb{E}(\#\{P^3 \subset G_p[E_y^{(0)}]\}) \le 8\omega^5 n p^3 = 8\omega^8 n^{-1/5},\end{aligned}$$

where $\#\{P^3 \subset G_p[E_y^{(0)}]\}$ denotes the number of copies of P^3 in $G_p[E_y^{(0)}]$. Now, from the independence of the events 'y is $E^{(0)}$-bad' ($y \in Y$), we have that the probability P_k that at least k such vertices y are $E^{(0)}$-bad satisfies

$$P_k \le \binom{n}{k}\left\{8\omega^8 n^{-1/5}\right\}^k \le \left(\frac{8e\omega^8 n^{4/5}}{k}\right)^k. \tag{11}$$

Thus, if $k = \lfloor 9e\omega^8 n^{4/5} \rfloor$, we have $P_k = o(1/n)$ with plenty to spare.

The above argument proves our lemma with '$E \subset E(G_p[\Gamma_{G_p}(x)])$' replaced by '$E = E(G_p[\Gamma_{G_p}(x)])$'. To complete the proof, we make the following simple observation. Suppose $x \in V(G_p)$ is fixed, $E \subset E^{(0)} = E(G_p[\Gamma_{G_p}(x)])$, and $y \in Y = V(G_p) \setminus (\{x\} \cup \Gamma_{G_p}(x))$. Then, y is necessarily $E^{(0)}$-bad whenever it is E-bad. Thus, an E-bad vertex $y \in \widetilde{W} = V(G_p) \setminus (V(E) \cup \{x\})$ is either contained in $\Gamma_{G_p}(x)$ or else it is $E^{(0)}$-bad. Since we may assume that $\Delta(G_p) \le 2pn = 2\omega n^{3/5} \le \omega^8 n^{4/5}$, our lemma follows. □

Lemma 11 above tells us that, for any x and any E, we have $\Delta(E_y) = \Delta(G_p[E_y]) \le 1$ for most $y \in \widetilde{W} = V(G_p) \setminus (\{x\} \cup V(E))$. Of course, since G_p is supposed not to contain H_{12}, we have $\Delta(E_y) \le 11$ for any vertex $y \in \widetilde{W}$.

For each vertex $y \in \widetilde{W}$, let $X_y = |E_y| = k_3(E, y)$ and let X'_y be the cardinality $\nu(E_y) = \nu(G_p[E_y])$ of a maximum matching in $G_p[E_y]$. Since $\Delta(E_y) \le 11$, we have $X'_y \ge X_y/2\Delta(E_y) \ge X_y/22$ for any $y \in \widetilde{W}$. Moreover, for any $y \in \widetilde{W}$ that is not E-bad, we clearly have $X'_y = X_y$.

Let us now fix $W \subset \widetilde{W}$. Put

$$X = X_W = \sum_{w \in W} X_w = \sum_{w \in W} k_3(E, w) = k_3(E, W),$$

and similarly $X' = X'_W = \sum_{w \in W} X'_w$. Let us write $\sum_{\mathrm{b}}$ for sum over all $w \in W$ that are E-bad and $\sum_{\mathrm{g}}$ for sum over all $w \in W$ that are not E-bad. Then

$$\begin{aligned} k_3(E, W) = X_W &= \sum\nolimits_{\mathrm{g}} X_w + \sum\nolimits_{\mathrm{b}} X_w \\ &= \sum\nolimits_{\mathrm{g}} X'_w + \sum\nolimits_{\mathrm{b}} X_w \le X'_W + \sum\nolimits_{\mathrm{b}} X_w. \end{aligned} \tag{12}$$

Now our aim is to bound the last two summands in (12).

For any two distinct vertices x and $y \in V(G_p)$, let Y_{xy} be the number of edges induced by the set $\Gamma_{G_p}(x, y) = \Gamma_{G_p}(x) \cap \Gamma_{G_p}(y)$ in G_p. Thus $Y_{xy} = |E(G_p[\Gamma_{G_p}(x, y)])|$, and $\mathbb{E}(Y_{xy}) = \binom{n-2}{2} p^5 = (1/2 + o(1))\omega^5$.

Lemma 12. *For almost every G_p we have $Y_{xy} \le 22\mathrm{e}^2 \max\{\log n, \omega^5\}$ for any pair of distinct vertices x, $y \in V(G_p)$.*

Proof. Let Y'_{xy} be the maximum cardinality $\nu(G_p[\Gamma_{G_p}(x, y)])$ of a matching in $G_p[\Gamma_{G_p}(x, y)]$. We claim that

$$\mathbb{P}\big\{Y'_{xy} \ge \mathrm{e}^2 \max\{\log n, \omega^5\}\big\} \le n^{-\mathrm{e}^2} = o(n^{-2}). \tag{13}$$

For convenience, put $\mu = \mathbb{E}(Y_{xy}) \le \omega^5$. By Lemma 2 in Janson **[10]**, for any $a \ge \mathrm{e}^2$, we have

$$\mathbb{P}(Y'_{xy} \ge a\mu) \le \exp\{-\mu(a \log a + 1 - a)\} \le \exp\left\{-\frac{1}{2} a (\log a) \mu\right\}. \tag{14}$$

We now check that (13) follows from (14). Suppose $\mu = \mathbb{E}(Y_{xy}) \le \log n$. Then we take $a = \mathrm{e}^2 \mu^{-1} \log n \ge \mathrm{e}^2$ in (14) and note that then

$$\mathbb{P}(Y'_{xy} \ge \mathrm{e}^2 \log n) = \mathbb{P}(Y'_{xy} \ge a\mu) \le \exp\{-\mathrm{e}^2 \log n\} = n^{-\mathrm{e}^2}.$$

Suppose now that $\mu = \mathbb{E}(Y_{xy}) > \log n$. Then we take $a = \mathrm{e}^2$ in (14) to obtain

$$\mathbb{P}(Y'_{xy} \ge \mathrm{e}^2 \omega^5) \le \mathbb{P}(Y'_{xy} \ge \mathrm{e}^2 \mu) \le \exp\{-\mathrm{e}^2 \log n\} = n^{-\mathrm{e}^2}.$$

Thus the claimed inequality (13) does hold. In particular, almost surely $Y'_{xy} \le \mathrm{e}^2 \max\{\log n, \omega^5\}$ for any distinct x, $y \in V(G_p)$. Our lemma now follows, since $\Delta(G_p[\Gamma_{G_p}(x, y)]) \le 11$, and hence $Y'_{xy} \ge Y_{xy}/22$. □

By Lemmas 11 and 12 above, an almost sure upper bound for the last summand in (12) is $220\mathrm{e}^3 \max\{\log n, \omega^5\} \omega^8 n^{4/5}$. Our aim now is to estimate X'_W. Fortunately, the method of proof of Lemma 2 in Janson **[10]** gives the following result immediately.

Lemma 13. *Suppose* $0 \le \varepsilon \le 1/2$. *Then for any* $W \subset \widetilde{W} = V(G_p) \setminus (\{x\} \cup V(E))$ *we have*

$$\mathbb{P}(X'_W \ge (1+\varepsilon)p^2|E||W|) \le \exp\left\{-\frac{1}{3}\varepsilon^2 p^2 |E||W|\right\}.$$

Proof. We follow an argument of Janson [**10**] (cf. the proof of Lemma 2 in [**10**]). Fix $W \subset \widetilde{W} = V(G_p) \setminus (\{x\} \cup V(E))$ and let $w \in W$. Let $\mathcal{F}$ be the family of all copies J of P^3 with middle vertex w and endvertices coinciding with endvertices of edges in E in the complete graph on $V(G_p)$. Thus each such J corresponds to a triangle determined by w and one edge from E. Let $\mathcal{F}_m$ ($m \ge 0$) be the collection of all m-tuples $(J_1, \dots, J_m)$ of pairwise edge-disjoint elements from $\mathcal{F}$. Let I_J be the indicator variable of the event that a fixed $J \in \mathcal{F}$ should be present in G_p. Thus $k_3(E, w) = X_w = \sum_{J \in \mathcal{F}} I_J$. We have

$$\begin{aligned}\mathbb{E}(\mathrm{e}^{tX'_w}) &= \mathbb{E}\left\{\sum_{m\ge 0}\binom{X'_w}{m}(\mathrm{e}^t - 1)^m\right\} \le \mathbb{E}\left\{\sum_{m \ge 0}\frac{1}{m!}\sum_{\mathcal{F}_m} I_{J_1}\dots I_{J_m}\left(\mathrm{e}^t-1\right)^m\right\}\\ &= \sum_{m\ge 0}\frac{1}{m!}\sum_{\mathcal{F}_m} p^{2m}\left(\mathrm{e}^t-1\right)^m \le \sum_{m\ge 0}\frac{1}{m!}\left\{\sum_{\mathcal{F}} p^2\left(\mathrm{e}^t - 1\right)\right\}^m = \mathrm{e}^{\lambda(\mathrm{e}^t-1)},\end{aligned}$$

where $\lambda = \mathbb{E}(X_w)$. Now, the X'_w ($w \in W$) are independent and therefore

$$\mathbb{E}(\mathrm{e}^{tX'_W}) = \mathbb{E}(\mathrm{e}^{t\sum_{w\in W} X'_w}) = \prod_{w \in W}\mathbb{E}(\mathrm{e}^{tX'_w}) \le \mathrm{e}^{\lambda|W|(\mathrm{e}^t-1)}.$$

Let $a \ge 1$. Note that then, by Markov's inequality, we have

$$\begin{aligned}\mathbb{P}(X'_W \ge a\lambda|W|) &= \mathbb{P}(\mathrm{e}^{tX'_W} \ge \mathrm{e}^{ta\lambda|W|})\\ &\le \mathbb{E}(\mathrm{e}^{tX'_W})\mathrm{e}^{-ta\lambda|W|} \le \mathrm{e}^{\lambda|W|(\mathrm{e}^t-1)}\mathrm{e}^{-ta\lambda|W|} = \mathrm{e}^{\lambda|W|(\mathrm{e}^t-1-at)}.\end{aligned}$$

Taking $t = \log a$, we have

$$\mathbb{P}(X'_W \ge a\lambda|W|) \le \mathrm{e}^{\lambda|W|(a-1-a\log a)} = \mathrm{e}^{-\lambda|W|(a\log a - a + 1)}.$$

Our lemma follows by setting $a = 1+\varepsilon$ in this last inequality. □

We are finally ready to prove Lemma 10.

Proof of Lemma 10. Take $C_0 = C_0(c_1, c_2, \theta) = \{12/(\theta^2 c_1 c_2)\}^{1/5}$. We proceed to show that this choice for C_0 will do. Thus let x, E, and W as in the statement of our lemma be given. To prove (10), it suffices to put together (12) and Lemmas 11, 12, and 13. Indeed, with the notation as above, by Lemma 13 we have

$$\mathbb{P}\left(X'_W \ge \left(1+\frac{\theta}{2}\right)p^2|E||W|\right) \le \exp\left\{-\frac{1}{3}\left(\frac{\theta}{2}\right)^2 p^2|E||W|\right\} \le \mathrm{e}^{-n},$$

where in the last inequality we used that $|E| \ge c_1 p^3 n^2$, $|W| \ge c_2 n$, and $C_0 = \{12/(\theta^2 c_1 c_2)\}^{1/5}$.

The number of choices for the triple (x, E, W) is at most $n\exp\{2\omega^3 n^{4/5}\log n\}2^n$. Indeed, it suffices to notice that we may assume that $|E| \le u_0 = \lfloor p^3n^2\rfloor = \lfloor\omega^3 n^{4/5}\rfloor$, and hence the number of choices for E is at most

$$\sum_{u\le u_0}\binom{n^2}{u} \le 2\binom{n^2}{u_0} \le \exp\{2\omega^3 n^{4/5}\log n\}.$$

Thus almost every G_p is such that $X'_W \le (1+\theta/2)p^2|E||W|$.

From (12) and Lemmas 11 and 12, we have, for almost every G_p,

$$k_3(E,W) \le \left(1+\frac{\theta}{2}\right)p^2|E||W| + 220\mathrm{e}^3\max\{\log n, \omega^5\}\omega^8 n^{4/5},$$

which is at most $(1+\theta)p^2|E||W|$, as required. □

In our next lemma we give a set of conditions that ensures that $k_3^{G_p}(E,W)$ ($E \subset E(G_p)$, $W \subset V(G_p)\setminus V(E)$) concentrates around its mean.

Lemma 14. *Suppose $\omega=\omega(n)\to\infty$ as $n\to\infty$. Let $0<p=p(n)\le 1$ with $p \ge \omega n^{-1/2}\log n$ be given. Then almost every $G_p = G_{n,p}$ is such that, for any $E \subset E(G_p)$ and $W \subset V(G_p)\setminus V(E)$ with $|E| \ge \omega np^{-1}\log n$ and $|W| \ge n/\log n$, we have*

$$k_3(E,W) = k_3^{G_p}(E,W) = (1+o(1))p^2|E||W|. \tag{15}$$

Proof. Let $\theta > 0$ be fixed. Let $M \subset E(K^n)$ be a matching in the complete graph K^n, and set $\nu = |M|$. Suppose that $\nu p^2/\log n \to \infty$ as $n\to\infty$. Let $W \subset V(G_p)\setminus V(M)$ be such that $w = |W| \ge n/\log n$. Let us write $k'_3(M,W) = k_3^{G_p\cup M}(M,W)$ for the number of triangles in $G_p \cup M$ that are determined by an edge of M and a vertex of W. Note that $k'_3(M,W)$ has binomial distribution with parameters $\nu w = |M||W|$ and p^2. Thus

$$k'_3(M,W) = (1+O_1(\theta))p^2|M||W| \tag{16}$$

with probability $1-\exp\{-\Omega(p^2\nu w)\}$. The number of matchings $M \subset E(K^n)$ of cardinality ν is no larger than $\binom{\binom{n}{2}}{\nu} \le n^{2\nu}$, and the number of sets W as above is no larger than $\binom{n}{w} \le n^w$. Hence, we see that (16) holds a.s. for any matching $M \subset E(K^n)$ with $|M| \ge \nu_0 = \omega(\log n)/4p^2$ and any $W \subset V(G_p)\setminus V(M)$ with $|W| \ge n/\log n$, since

$$\begin{aligned}\sum_{\nu\ge\nu_0}\sum_{w\ge n\log n} n^{2\nu+w}\exp\{-p^2\nu w\} &\le \sum_{\nu\ge\nu_0} n^{2\nu}\sum_{w\ge n\log n}\left(n\mathrm{e}^{-\Omega(p^2\nu)}\right)^w \\ &\le \sum_{\nu\ge\nu_0} n^{2\nu-\Omega(p^2\nu n/\log n)} \\ &= o(1).\end{aligned}$$

Thus, let us assume that our G_p does have this property. Clearly, we may also assume that $\Delta(G_p) \le 2pn$. We claim that under these assumptions our G_p necessarily satisfies (15) with '$o(1)$' replaced by '$O_1(\theta)$' for all E and W as in the statement of our lemma. To see this, fix $E \subset E(G_p)$ and $W \subset V(G_p)\setminus V(E)$ as in the lemma.

We shall now make use of the following simple fact that may be easily deduced from Vizing's theorem: if J is a graph of maximum degree $\Delta(J)$, then J admits a proper edge-colouring with at most $\Delta(J)+1$ colours such that the cardinality of any two colour classes differ by at most 1. Note that $\Delta(E)=\Delta(G_p[E])\leq\Delta(G_p)\leq 2pn$, and hence, by the observation above, we may write $E=E_1\cup\cdots\cup E_q$ where the E_i are matchings satisfying $\big||E_i|-|E_j|\big|\leq 1$ for all i and j, and moreover $q\leq\Delta(E)+1\leq 3pn$. In particular, $|E_i|\geq|E|/4pn\geq\nu_0$ for all i. Therefore (16) applies with $M=E_i$ and, since $k_3(E,W)=\sum_{1\leq i\leq q}k_3(E_i,W)$, our claim does hold. Lemma 14 follows by letting θ tend to 0. $\square$

§3. Pivotal Pairs of Vertices

3.1. A weighted Turán type result. Let H_* be a graph and $r\geq 3$ an integer. Let us write K^r_- for the graph with r vertices and $\binom{r}{2}-1$ edges, and let us say that its two vertices of degree $r-2$ are the *endvertices* of K^r_-. Let us say that the unordered pair $xy=\{x,y\}$ of distinct vertices of H_* is a *K^r_--connected* pair if there is a copy of K^r_- in H_* with endvertices x and y. Hence, if xy is a K^r_--connected pair of non-adjacent vertices, then the addition of xy to H_* creates a new copy of K^r in H_*. Thus, we shall also say that a K^r_--connected pair is *K^r-pivotal*, or simply *pivotal*. For technical reasons, let us also say that the vertex $x\in V(H_*)$ is by itself a K^r_--connected pair if x lies in a copy of K^{r-1} in H_*. The following is an asymptotic version of Turán's theorem for K^r.

(*) *Any graph H_* with k vertices and $e(H_*)$ edges contains at least*

$$(r-2)e(H_*)-(r-3)\binom{k+1}{2}$$

K^r_--connected pairs of vertices. $\square$

To check that (*) is indeed an asymptotic form of Turan's theorem, observe that if $\lambda=\lambda(H_*)=e(H_*)\binom{|H_*|}{2}^{-1}$ is the 'density' of $H_*=H_*^k$, then the lower bound in (*) for the number of K^r_--connected pairs in H_* is, for large k,

$$\sim\big((r-2)\lambda-r+3\big)\binom{k}{2}. \tag{17}$$

Note that (17) is bigger than $(1/(r-1))\binom{k}{2}$ for $\lambda>1-1/(r-1)$. Therefore we may deduce that any (large) H_* with $\lambda(H_*)>1-1/(r-1)$ necessarily contains a K^r, which is, of course, a weak form of Turán's theorem. Unfortunately, (*) does not seem to imply Turán's theorem for K^r in its precise form.

In the sequel we shall need, however, a weighted version of (*) for $r=4$. To describe this version we need some technical definitions. Also, to simplify the notation we restrict ourselves to the case $r=4$. (The general case does not present any further difficulty.) We start with a graph $H_*=H_*^k$ of order k and assume $\gamma=(\gamma_e)_{e\in E(H_*)}$ is an assignment of *weights* $\gamma_e\geq 0$ to the edges $e\in E(H_*)$ of H_*. Suppose $\mathbf{x}=(x_1,\dots,x_k)$ is an ordering of the vertices of H_*. For any two not necessarily distinct vertices x, $y\in V(H_*)$ that form a K^4_--connected pair in H_*, we let

$$w_{H_*,\gamma}(x,y)=\max\{\gamma_1\gamma_2\colon \exists z_1,z_2\in V(H_*)\text{ with }\gamma_i=\gamma_{z_iy}\ (i\in\{1,2\})\\ \text{and }xz_1,xz_2,z_1z_2,z_1y,z_2y\in E(H_*)\}.$$

For convenience, if x, y do not form a K^4_--connected pair, we put $w_{H_*,\gamma}(x,y)=0$. Let

$$w(H_*,\gamma,\mathbf{x})=\sum_{1\le i\le j\le k} w_{H_*,\gamma}(x_i,x_j)$$

and $\gamma(H_*)=\sum_{e\in E(H_*)}\gamma_e$. Our weighted version of (*) for $r=4$ is given in Lemma 15 below. We remark that to deduce the unweighted case (*) for $r=4$ from this lemma, it suffices to take $\bar\gamma=1$ and $\gamma_e=1$ for all edges $e\in E(H_*)$.

Lemma 15. *Let $H_*=H_*^k$ be a graph of order k and edge weights $\gamma=(\gamma_e)_{e\in E(H_*)}$ with $0\le\gamma_e\le\bar\gamma$ for all $e\in E(H_*)$, where $\bar\gamma\ge 1$. Then there is an ordering $\mathbf{x}=(x_1,\dots,x_k)$ of the vertices of H_* for which we have*

$$w(H_*,\gamma,\mathbf{x})\ge 2\gamma(H_*)-\bar\gamma\binom{k+1}{2}. \tag{18}$$

Proof. Our proof is by induction on k. Since the case $k\le 3$ is trivial, we assume that $k\ge 4$ and that our lemma holds for graphs H_* with at most 3 vertices. Note that if $\gamma(H_*)\le\bar\gamma k^2/4$, then (18) is trivially true, as in this case $2\gamma(H_*)-\bar\gamma\binom{k+1}{2}\le\bar\gamma(k^2/2-k(k+1)/2)=-\bar\gamma k/2\le 0$. Thus we may assume that $\gamma(H_*)>\bar\gamma k^2/4$. Since $\gamma_e\le\bar\gamma$ for all $e\in E(H_*)$, we have that H_* has more than $k^2/4$ edges. Therefore, there are three vertices y_1, y_2, and $y_3\in V(H_*)$ inducing a triangle in H_*. For $y\in V(H_*)$, let us write $d^\gamma(y)=d^{H_*,\gamma}(y)=\sum_{z\in\Gamma_{H_*}(y)}\gamma_{yz}$ for the γ-*degree* of y. We may assume that $d^\gamma(y_1)\le d^\gamma(y_2)\le d^\gamma(y_3)$. Put $x_1=y_1$, and by induction let $\mathbf{x}'=(x_2,\dots,x_k)$ be an ordering of the vertices of $H'_*=H_*-x_1=H_*-y_1$ such that

$$w(H'_*,\gamma,\mathbf{x}')\ge 2\gamma(H'_*)-\bar\gamma\binom{k}{2}.$$

For simplicity, γ above stands for the restriction $(\gamma_e)_{e\in E(H'_*)}$ of $\gamma=(\gamma_e)_{E(H_*)}$ to H'_*. Our aim now is to estimate $\sum_{1\le i\le k}w_{H_*,\gamma}(x_1,x_i)$ from below. For $j\in\{2,3\}$, set $\gamma_i(j)=\gamma_{y_jx_i}$ if $y_jx_i\in E(H_*)$ and set $\gamma_i(j)=0$ otherwise. Now observe that, since $\bar\gamma\ge 1$, for any reals $0\le\alpha\le\bar\gamma$ and $0\le\beta\le\bar\gamma$ we have $\alpha\beta\ge\alpha+\beta-\bar\gamma$. Therefore

$$\begin{aligned}\sum_{1\le i\le k}w_{H_*,\gamma}(x_1,x_i)&\ge\sum_{1\le i\le k}\gamma_i(2)\gamma_i(3)\ge\sum_{1\le i\le k}\big\{\gamma_i(2)+\gamma_i(3)-\bar\gamma\big\}\\&=d^\gamma(y_2)+d^\gamma(y_3)-\bar\gamma k\ge 2d^\gamma(y_1)-\bar\gamma k=2d^\gamma(x_1)-\bar\gamma k.\end{aligned}$$

Hence, with $\mathbf{x}=(x_1,x_2,\dots,x_k)$, we have

$$\begin{aligned}w(H_*,\gamma,\mathbf{x})&\ge\sum_{1\le i\le k}w_{H_*,\gamma}(x_1,x_i)+w_{H'_*,\gamma}(H'_*,\gamma,\mathbf{x}')\\&\ge 2d^\gamma(x_1)-\bar\gamma k+2\gamma(H'_*)-\bar\gamma\binom{k}{2}=2\gamma(H_*)-\bar\gamma\binom{k+1}{2},\end{aligned}$$

as required. □

3.2. Pivotal pairs in subgraphs of random graphs. In this section we turn to the study of K^4_--connected pairs in subgraphs of random graphs. Recall that given a graph H and two distinct vertices x, $y \in V(H)$, we say that the unordered pair $xy = \{x, y\}$ is a K^4_--connected pair if they are the endvertices of a copy of K^4_- in H, and that a single vertex by itself forms a K^4_--connected pair if it belongs to a triangle of H. Our main aim here is to prove Lemma 16 below, which roughly says that, if a subgraph $H \subset G_p = G_{n,p}$ of G_p is such that $e(H) \geq \lambda e(G_p)$ for some fixed $\lambda > 0$ and $p = p(n)$ is not too small, then the number of K^4_--connected pairs in H is, almost surely,

$$\gtrsim (2\lambda - 1)\binom{n}{2}.$$

This result is similar in spirit to assertion (*) in Section 3.1, but note that (*) applied to $H \subset G_p$ above gives nothing if $p = p(n) \to 0$ as $n \to \infty$. Lemma 16 below remedies this situation and recovers essentially the same lower bound for the number of K^4_--connected pairs.

In the sequel, for any given graph H, it will be convenient to define a graph $\Pi = \Pi_H$ on $V(H)$ by letting two distinct vertices x, $y \in V(H) = V(\Pi)$ be adjacent in Π if and only if they form a K^4_--connected in H.

Lemma 16. *Let a constant $0 < \sigma \leq 1$ be given. Then there is a constant $C_0 = C_0(\sigma)$ that depends only on σ for which the following holds. If $p = p(n) = \omega n^{-2/5}$ and $C_0 \leq \omega = \omega(n) = o(n^{1/190})$, then almost every $G_p = G_{n,p}$ is such that, for any subgraph $H \subset G_p$ of G_p, we have*

$$e(\Pi_H) \geq (2\lambda - 1 - \sigma)\binom{n}{2}, \tag{19}$$

where $\lambda = e(H)\left\{p\binom{n}{2}\right\}^{-1}$.

Before we proceed, let us remark that we shall apply Lemma 16 above with σ much smaller than λ. In fact, we shall be interested in the case in which λ is a little greater than $2/3$ and σ is very small.

The proof of Lemma 16 is based on the results of the previous three sections and on a further technical lemma, Lemma 17, which we now describe. The context in which Lemma 17 applies is as follows. Suppose $0 < \mu \leq 1/2$, $0 < \kappa \leq 1$, $0 < \delta' \leq 1$, $0 < \bar{\gamma}_0 \leq 1$, and $0 < \bar{\rho} \leq 1/4$ are constants, and let n and $m \geq \kappa n$ be integers. Let also $0 < p = p(n) \leq 1$ be given. Let $\bar{V}_1$, $\bar{V}_2$, $\bar{Z}_1$, and $\bar{Z}_2$ be pairwise disjoint sets of cardinality $(1-\mu)m \leq m_j = |\bar{V}_j| \leq m$ and $(1-\mu)m \leq m'_i = |\bar{Z}_i| \leq m$ ($i, j \in \{1,2\}$). Suppose $F = F_1 \cup F_2$ is a graph with F_j a tripartite graph with tripartition $V(F_j) = \bar{V}_j \cup \bar{Z}_1 \cup \bar{Z}_2$ ($j \in \{1,2\}$) and such that $E(F[\bar{Z}_1, \bar{Z}_2]) = E(F_j[\bar{Z}_1, \bar{Z}_2])$ for both $j \in \{1,2\}$. For simplicity, put $\gamma_Z = d_{F,p}(\bar{Z}_1, \bar{Z}_2)$ and let $\gamma_{ij} = d_{F,p}(\bar{Z}_i, \bar{V}_j)$ for i, $j \in \{1,2\}$. Suppose further that the following conditions hold:

(*i*) We have $\gamma_Z \geq \bar{\gamma}_0$ and $\gamma_{ij} \geq \bar{\gamma}_0$ for all i, $j \in \{1,2\}$.

(*ii*) At least $(1-\delta')m_1$ vertices x in $\bar{V}_1$ are such that

$$k_3^F(x) \geq (1 - \bar{\rho} - \delta')\gamma_{11}\gamma_{21}\gamma_Z p^3 m'_1 m'_2,$$

where $k_3^F(x)$ is the number of triangles of F that contain x.

(*iii*) At least $(1-\delta')e(F[\bar{Z}_1,\bar{Z}_2])$ edges e in $F[\bar{Z}_1,\bar{Z}_2]$ are such that

$$k_3^F(e,\bar{V}_2) \geq (1-\delta')\gamma_{12}\gamma_{22}p^2m_2, \tag{20}$$

where $k_3^F(e,\bar{V}_2)$ denotes the number of triangles in F determined by the edge e and some vertex in $\bar{V}_2$.

(*iv*) Let $c=\bar{\gamma}_0^3\kappa^2/8$. For any $x\in\bar{V}_1$, $E\subset E(F[\Gamma_F(x)])$, and $W\subset\bar{V}_2$ with $|E|\geq cp^3n^2$ and $|W|\geq cn$, we have

$$k_3^F(E,W) \leq \left(1+\frac{\delta'}{4}\right)p^2|E||W|.$$

(*v*) For all $E\subset E(F[\bar{Z}_1,\bar{Z}_2])$ with $|E|\geq pn^2/\log n$ and $W\subset\bar{V}_1$ with $|W|\geq n/\log n$, we have $k_3^F(E,W)\leq 2p^2|E||W|$.

We may now state a key technical lemma in the proof of Lemma 16.

Lemma 17. *Let constants $0<\bar{\gamma}_0\leq 1$ and $0<\sigma\leq 1$ be given. Then, with the notation above, if $\delta'\leq\delta_0'=\delta_0'(\bar{\gamma}_0,\sigma)=\sigma^2\bar{\gamma}_0/128$ and $\mu\leq\mu_0=\mu_0(\sigma)=\sigma/8$, then the number of K_-^4-connected pairs x_1x_2 with $x_j\in\bar{V}_j$ $(j\in\{1,2\})$ is at least $(1-\sigma)\gamma_{12}\gamma_{22}m^2$.*

Proof. Let us put $\alpha=\mu_0=\sigma/8$ and note that then $\delta_0'=\sigma^2\bar{\gamma}_0/128=\alpha^2\bar{\gamma}_0^2/2$. Let $0<\delta'\leq\delta_0$ and $0<\mu\leq\mu_0$ be given, and suppose that F is as described before our lemma.

In the sequel, an edge $e\in E(F[\bar{Z}_1,\bar{Z}_2])$ will be said to be $(\delta',K^3;F_2)$*-poor* if (20) fails. For $x\in\bar{V}_1$, let us write $k_3^{\mathrm{p}}(x)$ for the number of $(\delta',K^3;F_2)$-poor edges $e\in E(F[\bar{Z}_1,\bar{Z}_2])$ induced by the neighbourhood of x in F. Let us say that $x\in\bar{V}_1$ is *unusable* if $k_3^{\mathrm{p}}(x)\geq\alpha\gamma_{11}\gamma_{21}\gamma_Z p^3m_1'm_2'$. The proof is now split into a few claims.

Assertion 1. At most αm_1 vertices in $\bar{V}_1$ are unusable.

Suppose the contrary. Let us consider the number N of pairs (x,e) with x a vertex in $\bar{V}_1$ and e a $(\delta',K^3;F_2)$-poor edge in $E(F[\bar{Z}_1,\bar{Z}_2])$ such that there is a triangle of F containing both x and e. Since we are assuming that more than αm_1 vertices $x\in\bar{V}_1$ are unusable, we have

$$N>\alpha^2\gamma_{11}\gamma_{21}\gamma_Z p^3m_1'm_2'm_1\geq\alpha^2\bar{\gamma}_0^2\gamma_Z p^3m_1'm_2'm_1. \tag{21}$$

We now use condition (v) above to deduce that

$$N\leq 2p^2m_1\delta' e(F[\bar{Z}_1,\bar{Z}_2])=2\delta'\gamma_Z p^3m_1'm_2'm_1 \tag{22}$$

Comparing (21) and (22), we obtain that $\alpha^2\bar{\gamma}_0^2<2\delta'$, contradicting the fact that $\delta'\leq\delta_0'=\alpha^2\bar{\gamma}_0/2$. Thus Assertion 1 holds.

We now observe that condition (ii) above immediately implies the following.

Assertion 2. For at least $(1-2\alpha)m_1$ vertices x in $\bar{V}_1$, we have $k_3^F(x)-k_3^{\mathrm{p}}(x)\geq c_1p^3n^2$, where $c_1=(1-\bar{\rho}-2\alpha)(1-\mu)^2\kappa^2\bar{\gamma}_0^3\geq c$.

Now let $x\in\bar{V}_1$ be given. Let $E_x\subset E(F[\bar{Z}_1,\bar{Z}_2])$ be the set of edges e induced by the neighbourhood of x in F that are *not* $(\delta',K^3;F_2)$-poor. Thus $|E_x|=k_3^F(x)-k_3^{\mathrm{p}}(x)$. Let

$$\begin{aligned} W_x&=\{y\in\bar{V}_2\colon E(F[\Gamma_F(x)])\cap E(F[\Gamma_F(y)])\neq\emptyset\}\\ &=\{y\in\bar{V}_2\colon xy \text{ is a } K_-^4\text{-connected pair}\}. \end{aligned}$$

Assertion 3. For at least $(1-2\alpha)m_1$ *vertices* $x \in \bar{V}_1$, *we have* $|W_x| \geq (1-\sigma/2)\gamma_{12}\gamma_{22}m_2$.

Let $x \in \bar{V}_1$ be such that $|E_x| = k_3^F(x) - k_3^{\mathrm{p}}(x) \geq c_1 p^3 n^2$, and suppose that $|W_x| < (1-\sigma/2)\gamma_{12}\gamma_{22}m_2$. We now use (iv) above to deduce that

$$(1-\delta')\gamma_{12}\gamma_{22}p^2 m_2|E_x| \leq k_3^F(E_x, W_x) \leq \left(1+\frac{\delta'}{4}\right)p^2|E_x||W_x|$$
$$\leq \left(1+\frac{\delta'}{4}\right)\left(1-\frac{\sigma}{2}\right)\gamma_{12}\gamma_{22}p^2 m_2|E_x|,$$

which is a contradiction since $1-\delta' > (1+\delta'/4)(1-\sigma/2)$. Thus any $x \in \bar{V}_1$ with $|E_x| = k_3^F(x) - k_3^{\mathrm{p}}(x) \geq c_1 p^3 n^2$ is such that $|W_x| \geq (1-\sigma/2)\gamma_{12}\gamma_{22}m_2$, and hence Assertion 3 follows from Assertion 2.

From Assertion 3, we deduce that at least

$$(1-2\alpha)\left(1-\frac{\sigma}{2}\right)\gamma_{12}\gamma_{22}m_1m_2$$
$$\geq (1-2\alpha)(1-\mu)^2\left(1-\frac{\sigma}{2}\right)\gamma_{12}\gamma_{22}m^2 \geq (1-\sigma)\gamma_{12}\gamma_{22}m^2$$

pairs x_1x_2 ($x_j \in \bar{V}_j$, $j \in \{1,2\}$) are K^4_--connected with respect to F, thereby proving Lemma 17. □

We are now ready to prove Lemma 16. Our proof will make use of several of our previous lemmas.

Proof of Lemma 16. Let $0 < \sigma \leq 1$ be given. We now define the many constants with which we shall apply Lemmas 4, 5, 7, 9, 10, 14, 15, and 17.

Let $\gamma_0 = \sigma/100$, $\bar{\gamma}_0 = \gamma_0/2$, $\alpha = \gamma_0/24\mathrm{e}$ and $k_0 = \lceil 100/\sigma \rceil$. Let $\delta = (\delta')^2/2$, where $\delta' = \sigma^3/2\times 10^6 \leq \delta_0' = \delta_0'(\bar{\gamma}_0, \sigma/7) = \sigma^2\bar{\gamma}_0/6272$. Note that $\delta_0'(\bar{\gamma}_0, \sigma/7)$ is as given in Lemma 17. We now let

$$\varepsilon = \min\left\{10^{-16}\sigma^6\gamma_0, \frac{1}{2}\varepsilon_0(\alpha, \bar{\gamma}_0, \delta)\right\},$$

where $\varepsilon_0 = \varepsilon_0(\alpha, \bar{\gamma}_0, \delta)$ is as defined in Lemma 7. Put $\bar{\varepsilon} = 2\varepsilon$, $\rho = 2\varepsilon/\gamma_0$ and $\bar{\rho} = 5\varepsilon/\gamma_0$. For later reference, note that

$$\bar{\rho} = \frac{5\varepsilon}{\gamma_0} \leq 5\times 10^{-16}\sigma^6 \leq \frac{\delta}{27}, \tag{23}$$

and if $k \geq k_0$, then, with plenty to spare, we have

$$\frac{2\sigma}{7(k-1)} + \frac{6}{k-1} \leq \frac{\sigma}{7}, \tag{24}$$

and

$$\frac{3\varepsilon}{\gamma_0} + \frac{1}{k} \leq \frac{2}{7}\sigma. \tag{25}$$

Set $\mu = 6\varepsilon \leq \mu_0(\sigma/7)$, where $\mu_0(\sigma/7) = \sigma/56$ is as given in Lemma 17. Now let $K_0 = K_0(\varepsilon, k_0)$ and $\eta = \eta(\varepsilon, k_0)$ be as given by Lemma 4. We may assume that $\eta \leq \min\{\sigma/7, \varepsilon/2K_0\}$. Put $\kappa = 1/2K_0$, and let $c = \bar{\gamma}_0^3\kappa^2/8$. Finally, let $C_0 = C_0(c, c, \delta'/4)$, where $C_0(c, c, \delta'/4)$ is as given by Lemma 10. We claim that this choice of $C_0 = C_0(\sigma)$ will do in Lemma 16, and proceed to prove this assertion.

Let $p = p(n) = \omega n^{-2/5}$, where $C_0 \leq \omega = \omega(n) = o(n^{1/190})$. Let us consider the following conditions for $G_p = G_{n,p}$.

(a) G_p is η-upper-uniform and has size $e(G_p) = (1 + o(1))p\binom{n}{2}$.
(b) Suppose $m \geq \kappa n$. Then, for any $T \geq (3/4)\bar{\gamma}_0 pm^2$ and any $\mathbf{m} = (m_1, m_2, m_3)$ with $m/2 \leq m_i \leq m$ ($i \in \{1, 2, 3\}$), our G_p contains no copy of any graph $F \in \mathcal{F}_p^\delta(\bar{\varepsilon}, \bar{\gamma}_0, \bar{\rho}; \mathbf{m}, T)$ as a subgraph.
(c) Inequality (10) in Lemma 10 holds for all E and W as in the statement of that lemma with $c_1 = c_2 = c$ and $\theta = \delta'/4$.
(d) Relation (15) in Lemma 14 holds for all $E \subset E(G_p)$ and $W \subset V(G_p) \setminus V(E)$ with $|E| \geq pn^2/\log n$ and $|W| \geq n/\log n$.

Claim. Conditions (a)–(d) *hold for almost every* G_p.

Proof of the Claim. Condition (a) clearly holds almost surely. We use Lemma 7 to prove that (b) holds almost surely as well. Let $\mathbf{m}$, m, and T be as in (b). Note that the number of choices for $\mathbf{m}$ and m is trivially at most n^4. The number of copies of a fixed graph $F \in \mathcal{F}_p^\delta(\bar{\varepsilon}, \bar{\gamma}_0, \bar{\rho}; \mathbf{m}, T)$ that the complete graph K^n on n vertices contains is clearly at most n^n. Thus, since $\bar{\rho} \leq \delta/27$ (see (23)) and $\bar{\varepsilon} = 2\varepsilon \leq \varepsilon_0(\alpha, \bar{\gamma}_0, \delta)$, Lemma 7 applies to give that the expected number of copies of elements from $\mathcal{F}_p^\delta(\bar{\varepsilon}, \bar{\gamma}_0, \bar{\rho}; \mathbf{m}, T)$ that G_p contains is, for sufficiently large n, at most

$$n^{4+n}\big|\mathcal{F}_p^\delta(\bar{\varepsilon}, \bar{\gamma}_0, \bar{\rho}; \mathbf{m}, T)\big|p^T \leq n^{4+n}\alpha^T\binom{3m^2}{T}p^T \leq n^{4+n}\left(\frac{3\mathrm{e}\alpha pm^2}{T}\right)^T,$$

which, since $T \geq (3/4)\bar{\gamma}_0 pm^2$, is at most

$$n^{4+n}\left(\frac{8\mathrm{e}\alpha}{\gamma_0}\right)^T \leq n^{4+n}3^{-T} \leq 2^{-T}.$$

Summing over all $T \geq (3/4)\bar{\gamma}_0 pm^2$, we obtain that ($b$) holds almost always. Condition (c) holds almost surely for G_p by Lemma 10 and the choice of C_0. To check that (d) holds for a.e. G_p, by Lemma 14 it suffices to see that $pn^{1/2}/\log n \to \infty$ and that $(pn^2/\log n)/np^{-1}\log n \to \infty$ as $n \to \infty$. This completes the proof of our claim.

In the remainder of the proof, we show that (19) holds for any subgraph $H \subset G_p$ whenever G_p satisfies conditions (a)–(d) above. This clearly proves our lemma. Thus let us assume that $H \subset G_p$ is given and that G_p satisfies (a)–(d). We may clearly assume that H is a spanning subgraph of G_p.

Let us apply Lemma 4 to the η-upper-uniform graph H, with parameters ε and k_0. Let $V(H) = V_0 \cup \cdots \cup V_k$ be the (ε, H, p)-regular (ε, k)-equitable partition we obtain in this way. Let H' be the subgraph of H on $\bigcup_{1\leq i\leq k} V_i$ with $e \in E(H)$ an edge of H' if and only if e joins two vertices that belong to distinct classes of our regular partition, say, V_i and V_j ($1 \leq i < j \leq k$), with (V_i, V_j) an (ε, H, p)-regular pair and $d_{H,p}(V_i, V_j) \geq \gamma_0$. In the sequel, we let m stand for the common cardinality of the sets V_i ($1 \leq i \leq k$). Recall that $\lambda = e(H)\left\{p\binom{n}{2}\right\}^{-1}$. We now check the following simple fact.

Assertion 1. We have $e(H') \geq (\lambda - \sigma/7)p\binom{n}{2}$ *provided* n *is sufficiently large.*

From the η-upper-uniformity of H it follows that if $W \subset V = V(H)$, $|W| \geq 2\eta n$ then $e(H[W]) \leq (1+\eta)p\binom{|W|}{2}$. Hence $e(H[V_0]) \leq (3/5)\varepsilon^2 pn^2$. Also, $e_H(V_0, V \setminus V_0)$ is at most $(1+\eta)p|V_0||V \setminus V_0| \leq (6/5)\varepsilon pn^2$. Thus the number of edges of H incident to V_0 is at most $4\varepsilon p\binom{n}{2}$ for large enough n. Now note that $\sum e_H(V_i, V_j)$ with the sum over all $1 \leq i < j \leq k$ such that (V_i, V_j) is *not* (ε, H, p)-regular is at most $\varepsilon\binom{k}{2}(1+\eta)pm^2 \leq 2\varepsilon p\binom{n}{2}$. Also, $\sum e_H(V_i, V_j)$ with the sum extended over all $1 \leq i < j \leq k$ such that $d_{H,p}(V_i, V_j) \leq \gamma_0$ is at most $(1+\eta)\gamma_0\binom{k}{2}pm^2 \leq 2\gamma_0 p\binom{n}{2}$. Finally, we have that $\sum_{1\leq i\leq k} e(H[V_i]) \leq k(1+\eta)p\binom{m}{2} \leq (2p/k)\binom{n}{2}$. Therefore $|E(H) \setminus E(H')| \leq (6\varepsilon + 2/k + 2/\gamma_0)p\binom{n}{2} \leq (\sigma/7)p\binom{n}{2}$ if n is sufficiently large, as required.

We now define a graph H_* on $[k] = \{1, \ldots, k\}$ by letting ij $(1 \leq i < j \leq k)$ be an edge of H_* if and only if (V_i, V_j) is an (ε, H, p)-regular pair and $d_{H,p}(V_i, V_j) \geq \gamma_0$. We write γ_{ij} for $d_{H,p}(V_i, V_j)$ for all $ij \in E(H_*)$, and put $\gamma = (\gamma_e)_{e\in E(H_*)}$. Now put $\bar{\gamma} = 1 + \sigma/7$, and notice that then the definition of η and (a) above gives that $\gamma_e \leq 1 + \eta \leq \bar{\gamma}$ for all $e \in E(H_*)$.

Lemma 15 now tells us that, suitably adjusting the notation, the ordering $\mathbf{x} = (1, \ldots, k)$ of the vertices of H_* is such that

$$w(H_*, \gamma, \mathbf{x}) = \sum_{1\leq i\leq j\leq k} w_{H_*,\gamma}(i,j) \geq 2\gamma(H_*) - \bar{\gamma}\binom{k+1}{2}. \tag{26}$$

In our next assertion we bound $\gamma(H_*)$.

Assertion 2. We have $\gamma(H_*) = \sum_{e\in E(H_*)} \gamma_e \geq (\lambda - \sigma/7)\binom{k}{2}$.

Indeed, we have $e(H') = \sum_{e\in E(H_*)} \gamma_e pm^2 = \gamma(H_*)pm^2$, and hence, by Assertion 1,

$$\gamma(H_*) \geq \frac{1}{m^2}\left(\lambda - \frac{\sigma}{7}\right)\binom{n}{2} \geq \left(\lambda - \frac{\sigma}{7}\right)\binom{n}{2}\Big/\frac{k^2}{n^2} \geq \left(\lambda - \frac{\sigma}{7}\right)\binom{k}{2}.$$

Our next step relates the number of K^4_--connected pairs meeting two fixed classes V_i and V_j $(1 \leq i < j \leq k)$ with the summand $w_{H_*,\gamma}(i,j)$ appearing in (26).

Assertion 3. Suppose ι_1, $\iota_2 \in [k]$ *are two distinct vertices of* H_*. *Then the number of* K^4_-*-connected pairs* x_1x_2 *with* $x_j \in V_{\iota_j}$ $(j \in \{1,2\})$ *is at least*

$$(1 - \sigma/7 - 2\varepsilon/\gamma_0)w_{H_*,\gamma}(\iota_1, \iota_2)m^2.$$

The above assertion is an easy consequence of Lemma 17, although we shall have to work a little to check that that lemma does apply here.

Let us start by observing that, trivially, if ι_1 and ι_2 are not K^4_--connected in H_*, then by definition $w_{H_*,\gamma}(\iota_1, \iota_2) = 0$, and hence there is nothing to prove. Thus let us assume that this is not the case, and let ι_3, $\iota_4 \in [k]$ be two vertices of H_* such that $\iota_1\iota_3$, $\iota_1\iota_4$, $\iota_3\iota_4$, $\iota_2\iota_3$, $\iota_2\iota_4 \in E(H_*)$. Choosing ι_3 and ι_4 suitably, we may further assume that $w_{H_*,\gamma}(\iota_1, \iota_2) = \gamma_{\iota_2\iota_3}\gamma_{\iota_2\iota_4}$. We may now restrict our attention to the 4-partite subgraph of H induced by the V_{ι_a} $(1 \leq a \leq 4)$.

Let $J = H[V_{\iota_1}, V_{\iota_2}, V_{\iota_3}, V_{\iota_4}]$ and write L for the graph on $[4] = \{1, 2, 3, 4\}$ isomorphic to K^4_-, with 1 and 2 as the endvertices. We first apply Lemma 5 to J to obtain $\bar{U}_{\iota_a} \subset V_{\iota_a}$ $(1 \leq a \leq 4)$ such that the following holds. Putting $m_a = |\bar{U}_{\iota_a}|$, we have $(1-\mu)m \leq m_a \leq m$ for all $1 \leq a \leq 4$ and, furthermore, if $ab \in E(L)$ and $x \in \bar{U}_{\iota_a}$, then

$$d_{ab}(x) = |\Gamma_J(x) \cap \bar{U}_{\iota_b}| = (1 + O_1(\rho))d_{J,p}(V_{\iota_a}, V_{\iota_b})pm. \tag{27}$$

Let $F = J[\bar{U}_{\iota_1}, \bar{U}_{\iota_2}, \bar{U}_{\iota_3}, \bar{U}_{\iota_4}]$. Since $d_{J,p}(V_{\iota_a}, V_{\iota_b}) = d_{F,p}(\bar{U}_{\iota_a}, \bar{U}_{\iota_b}) + O_1(\varepsilon)$ and $d_{F,p}(\bar{U}_{\iota_a}, \bar{U}_{\iota_b}) \geq \gamma_0 - \varepsilon \geq \bar{\gamma}_0 = \gamma_0/2$, we have

$$d_{J,p}(V_{\iota_a}, V_{\iota_b}) = (1 + O_1(2\varepsilon/\gamma_0))d_{F,p}(\bar{U}_{\iota_a}, \bar{U}_{\iota_b}).$$

Moreover, as $(1-\mu)m \leq m_b \leq m$, we have $m = (1+O_1(2\mu))m_b$. Thus, relation (27) gives that, for any $x \in \bar{U}_{\iota_a}$,

$$d_{ab}(x) = (1 + O_1(\bar{\rho}))d_{F,p}(\bar{U}_{\iota_a}, \bar{U}_{\iota_b})pm_b,$$

where, as defined above, $\bar{\rho} = 5\varepsilon/\gamma_0$. Note also that $\bar{\gamma}_0 \leq d_{F,p}(\bar{U}_{\iota_a}, \bar{U}_{\iota_b}) \leq \bar{\gamma} = 1 + \sigma/7$ (cf. condition (a) above).

Our immediate aim now is to apply Lemma 17. To make our current notation the same as the one used in that lemma, let us put $\bar{V}_j = \bar{U}_{\iota_j}$ and $\bar{Z}_j = \bar{U}_{2+j}$ for $j \in \{1,2\}$ and $F_j = J[\bar{V}_j, \bar{Z}_1, \bar{Z}_2]$ $(j \in \{1,2\})$. Clearly, $F = J[\bar{V}_1, \bar{V}_2, \bar{Z}_1, \bar{Z}_2] = F_1 \cup F_2$ and $F[\bar{Z}_1, \bar{Z}_2] = F_1[\bar{Z}_1, \bar{Z}_2] = F_2[\bar{Z}_1, \bar{Z}_2]$. Also, let $m_j = |\bar{V}_j|$ and $m_i' = |\bar{Z}_i|$ $(i, j \in \{1,2\})$.

Let $\mathbf{m}_j = (m_j, m_1', m_2')$ and $T_j = e(F_j) \geq (3/4)\bar{\gamma}_0 pm^2$ $(j \in \{1,2\})$. Observe that by (b) above we may conclude that

$$F_j = H[\bar{V}_j, \bar{Z}_1, \bar{Z}_2] \in \mathcal{F}_p(\bar{\varepsilon}, \bar{\gamma}_0, \bar{\rho}; \mathbf{m}_j, T_j) \setminus \mathcal{F}_p^\delta(\bar{\varepsilon}, \bar{\gamma}_0, \bar{\rho}; \mathbf{m}_j, T_j)$$

for both $j \in \{1,2\}$. We shall now invoke Lemma 17, but to see that that lemma does apply we verify the following claim.

Claim. Conditions (i)–(v) given before the statement of Lemma 17 hold.

Proof of the Claim. Condition (i) has already been seen to hold. To see that (ii) holds, we simply apply Lemma 9 to the (δ, K^3)-balanced graph F_1. Now, condition (iii) clearly holds as F_2 is (δ, K^3)-balanced. Finally, condition (iv) is equivalent to (c), while condition (v) follows from (d). This finishes the proof of our claim.

In view of the above claim and the definitions of $\delta_0' = \delta_0'(\bar{\gamma}_0, \sigma/7)$ and $\mu_0 = \mu_0(\sigma/7)$, we see that we may indeed apply Lemma 17 to deduce that the number of K^4_--connected pairs x_1x_2 with $x_j \in V_{\iota_j}$ $(j \in \{1,2\})$ is at least

$$\left(1 - \frac{\sigma}{7}\right) d_{F,p}(\bar{Z}_1, \bar{V}_2) d_{F,p}(\bar{Z}_2, \bar{V}_2) m^2. \tag{28}$$

We now use that, for $i \in \{1,2\}$, we have

$$d_{F,p}(\bar{Z}_i, \bar{V}_2) \geq d_{H,p}(V_{\iota_2}, V_{\iota_{i+2}}) - \varepsilon \geq \left(1 - \frac{\varepsilon}{\gamma_0}\right) d_{H,p}(V_{\iota_2}, V_{\iota_{i+2}}),$$

since $d_{H,p}(V_{\iota_2}, V_{\iota_{i+2}}) \geq \gamma_0$. Thus the quantity in (28) is at least

$$\left(1 - \frac{\sigma}{7}\right)\left(1 - \frac{\varepsilon}{\gamma_0}\right)^2 d_{H,p}(V_{\iota_2}, V_{\iota_3}) d_{H,p}(V_{\iota_2}, V_{\iota_4}) m^2$$
$$\geq \left(1 - \frac{\sigma}{7} - \frac{2\varepsilon}{\gamma_0}\right) \gamma_{\iota_2\iota_3}\gamma_{\iota_2\iota_4} m^2,$$

which concludes the proof Assertion 3, since ι_3 and ι_4 were chosen so as to have $w_{H_*,\gamma}(\iota_1,\iota_2) = \gamma_{\iota_2\iota_3}\gamma_{\iota_2\iota_4}$.

The proof of our lemma is completed in the next assertion. Recall that Π_H stands for the graph on $V(H)$ with two vertices of H adjacent in Π_H if and only if they are K^4_--connected in H.

Assertion 4. We have $e(\Pi_H) \geq (2\lambda - 1 - \sigma)\binom{n}{2}$.

By Assertion 3 we have that

$$e(\Pi_H) \geq \sum_{1\leq i<j\leq k} e(\Pi_H[V_i, V_j]) \geq \sum_{1\leq i<j\leq k} \left(1 - \frac{\sigma}{7} - \frac{2\varepsilon}{\gamma_0}\right) w_{H_*,\gamma}(i,j) m^2. \tag{29}$$

Now, clearly, we have

$$\begin{aligned}\sum_{1\leq i<j\leq k} w_{H_*,\gamma}(i,j) &= w(H_*,\gamma,\mathbf{x}) - \sum_{1\leq i\leq k} w_{H_*,\gamma}(i,i)\\ &\geq w(H_*,\gamma,\mathbf{x}) - \bar\gamma^2 k \geq w(H_*,\gamma,\mathbf{x}) - 2k,\end{aligned}$$

since $\bar\gamma^2 = (1+\sigma/7)^2 \leq 2$. Thus, recalling (26) and using that $m \geq (1-\varepsilon)n/k$, we have from (29) that $e(\Pi_H)$ is at least

$$\begin{aligned}&\left(1 - \frac{\sigma}{7} - \frac{2\varepsilon}{\gamma_0}\right)(1-\varepsilon)^2\frac{n^2}{k^2}\Bigl(w(H_*,\gamma,\mathbf{x}) - 2k\Bigr)\\ &\qquad\geq \left(1 - \frac{\sigma}{7} - \frac{3\varepsilon}{\gamma_0}\right)\frac{n^2}{k^2}\left(2\gamma(H_*) - \bar\gamma\binom{k+1}{2} - 2k\right),\end{aligned}$$

which, by Assertion 2, is at least

$$\left(1 - \frac{\sigma}{7} - \frac{3\varepsilon}{\gamma_0}\right)\left(1 - \frac{1}{k}\right)\left(2\left(\lambda - \frac{\sigma}{7}\right) - \left(1 + \frac{\sigma}{7}\right)\binom{k+1}{2}\binom{k}{2}^{-1} - 2k\binom{k}{2}^{-1}\right)\binom{n}{2},$$

which is, as one may check using (24) and (25), at least

$$\left(1 - \frac{\sigma}{7} - \frac{3\varepsilon}{\gamma_0} - \frac{1}{k}\right)\left(2\lambda - 1 - \frac{4\sigma}{7}\right)\binom{n}{2} \geq (2\lambda - 1 - \sigma)\binom{n}{2},$$

proving Assertion 4.

The proof of Lemma 16 is complete. □

§4. Proof of the Main Result

We first prove Theorem 2 under the extra hypothesis that $p = p(n)$ should not be too large. More precisely, we prove the following result. Recall that we write $G \to_\gamma H$ if any subgraph J of G with $e(J) \geq \gamma e(G)$ contains a copy of H.

Lemma 18. *Let a constant $\eta > 0$ be given. Then there is a constant $C = C(\eta)$ that depends only on η for which the following holds. If $0 \leq p = p(n) = \omega n^{-2/5} \leq 1$ and $C \leq \omega = \omega(n) = o(n^{1/190})$, then almost every $G_p = G_{n,p}$ is such that $G_p \to_{2/3+\eta} K^4$.*

Proof. Let $\varepsilon = \sigma = \eta/10$ and $k = \lceil 1 + 24(\log 2)\eta^{-2}\rceil$. Let $C = kC_0(\sigma)$, where $C_0(\sigma)$ is as given in Lemma 16. We shall show that this choice of $C = C(\eta)$ will do in our result. Thus let $p = p(n)$ be as in the statement of Theorem 2, and consider the space $\mathcal{G}(n,p)$ of the random graphs $G_{n,p} = G_p$. In this proof, we shall write G_p as a union of sparser, independent random graphs.

Let p_1 be such that $1-p = (1-p_1)^k$, and note that then $p/k \leq p_1 = (1+o(1))p/k$. Put $\Omega = \prod_{1\leq j\leq k} \mathcal{G}(n,p_1)$. We shall write $\mathbf{G} = (G_{p_1}^{(1)}, \dots, G_{p_1}^{(k)})$ for a general random element of Ω. Thus the $G_{p_1}^{(j)}$ $(1 \leq j \leq k)$ are independent random graphs, each taken from $\mathcal{G}(n,p_1)$. For any given $\mathbf{G} = (G_{p_1}^{(1)}, \dots, G_{p_1}^{(k)}) \in \Omega$, let us put $G_p = G_p(\mathbf{G}) = G_{p_1}^{(1)} \cup \dots \cup G_{p_1}^{(k)}$, and note that then the map $\mathbf{G} = (G_{p_1}^{(j)})_{1\leq j\leq k} \in \Omega \mapsto G_p = G_p(\mathbf{G}) \in \mathcal{G}(n,p)$ is measure-preserving. We may thus study $\mathcal{G}(n,p)$ investigating the random elements $\mathbf{G} \in \Omega$. Let us define $\Omega' \subset \Omega$ by letting $\mathbf{G} = (G_{p_1}^{(j)})_{1\leq j\leq k} \in \Omega$ belong to Ω' if and only if (i) $e(G_p) = (1 + O_1(\varepsilon))p\binom{n}{2}$, (ii) $e(G_{p_1}^{(j)}) = (1 + O_1(\varepsilon))(p/k)\binom{n}{2}$ for all $1 \leq j \leq k$ and, finally, (iii) for all $1 \leq j \leq k$, the graph $G_{p_1}^{(j)}$ has the property that if $E \subset E(G_{p_1}^{(j)})$ and $\lambda = |E| \left\{p_1\binom{n}{2}\right\}^{-1}$, then

$$e(\Pi_E) \geq (2\lambda - 1 - \sigma)\binom{n}{2}. \tag{30}$$

For simplicity, Π_E above stands for Π_H, where $H = H(E)$ is the graph on $V(G_p)$ with edge set E.

Elementary facts concerning random graphs and Lemma 16 gives that $\mathbb{P}(\Omega') = 1 - o(1)$. In the sequel, we shall often condition on Ω' and we shall write $\mathbb{P}'(A)$ for the conditional probability $\mathbb{P}(A \mid \Omega')$ for any event $A \subset \Omega$.

Let $\mathcal{B} \subset \Omega$ be the set of $\mathbf{G} = (G_{p_1}^{(1)}, \dots, G_{p_1}^{(k)})$ that admit a set $F \subset E(G_p)$ with $|F| \geq (2/3 + \eta)e(G_p)$ but $G_p[F] \not\supset K^4$. We need to show that $\mathbb{P}(\mathcal{B}) = o(1)$, or, equivalently, that $\mathbb{P}'(\mathcal{B}) = o(1)$. Let us put $\mathcal{B}' = \mathcal{B} \cap \Omega'$. For each $\mathbf{G} \in \mathcal{B}$, let us fix once and for all a set $F = F(\mathbf{G}) \subset E(G_p)$ as required in the definition of $\mathcal{B}$. Let us also put $F^{(j)} = F^{(j)}(\mathbf{G}) = F \cap E(G_{p_1}^{(j)})$ $(1 \leq j \leq k)$ and set $f = f(\mathbf{G}) = |F|$ and $f^{(j)} = f^{(j)}(\mathbf{G}) = |F^{(j)}|$ $(1 \leq j \leq k)$.

Now let $\gamma^{(j)} = \gamma^{(j)}(\mathbf{G}) = f^{(j)}/p_1\binom{n}{2}$, and note that then we have $f \leq f^{(1)} + \dots + f^{(k)}$, and hence that

$$\frac{2}{3} + \eta \leq \gamma^{(1)}\frac{p_1\binom{n}{2}}{e(G_p)} + \dots + \gamma^{(k)}\frac{p_1\binom{n}{2}}{e(G_p)} \leq \frac{1}{k}\left(\frac{1+\varepsilon}{1-\varepsilon}\right)\left(\gamma^{(1)} + \dots + \gamma^{(k)}\right),$$

from which we conclude that

$$\gamma^* = \gamma^*(\mathbf{G}) = \max_{1\leq j\leq k} \gamma^{(j)} \geq \operatorname*{Ave}_{1\leq j\leq k} \gamma^{(j)} \geq \frac{1-\varepsilon}{1+\varepsilon}\left(\frac{2}{3} + \eta\right) \geq \frac{2}{3}(1+\eta), \tag{31}$$

where the last inequality follows from the choice of ε. Let us also note that

$$\gamma^* \leq 1 + \varepsilon, \tag{32}$$

since we are assuming that (ii) above holds. For each $1 \leq j \leq k$, let

$$\mathcal{B}'_j = \{\mathbf{G} \in \mathcal{B}' \colon \gamma^{(j)}(\mathbf{G}) = \gamma^*(\mathbf{G})\}.$$

Clearly $\mathcal{B}' = \bigcup_{1\le j\le k} \mathcal{B}'_j$, and hence it suffices to show that $\mathbb{P}'(\mathcal{B}'_j) = o(1)$ for all j. Thus we now *fix* $j \in [k]$ and proceed to show that $\mathcal{B}'_j$ almost surely does not hold. We have

$$\begin{aligned}\mathbb{P}'(\mathcal{B}'_j) &= \sum_{G_0} \mathbb{P}'(\mathcal{B}'_j \cap \{\mathbf{G}\in\Omega : G^{(j)}_{p_1} = G_0\}) \\ &= \sum_{G_0} \mathbb{P}'(\mathcal{B}'_j \mid G^{(j)}_{p_1} = G_0)\mathbb{P}'(G^{(j)}_{p_1} = G_0) \\ &\le \max_{G_0} \mathbb{P}'(\mathcal{B}'_j \mid G^{(j)}_{p_1} = G_0), \end{aligned} \tag{33}$$

where G_0 ranges over all graphs on $V(G_p)$ with $e(G_0) = (1+O_1(\varepsilon))(p/k)\binom{n}{2}$ and such that (30) holds for all $E \subset E(G_0)$ with $\lambda = |E|/p_1\binom{n}{2}$. We now *fix* one such G_0 and proceed to show an upper bound for (33). For each $F_0 \subset E(G_0)$, let

$$P'(j,G_0,F_0) = \mathbb{P}'(\mathbf{G}\in\mathcal{B}'_j,\ F^{(j)} = F_0 \mid G^{(j)}_{p_1} = G_0). \tag{34}$$

Then

$$\mathbb{P}'(\mathcal{B}'_j \mid G^{(j)}_{p_1} = G_0) = \sum_{F_0\subset E(G_0)} P'(j,G_0,F_0) \le 2^{(1+\varepsilon)\frac{p}{k}\binom{n}{2}} \max_{F_0\subset E(G_0)} P'(j,G_0,F_0). \tag{35}$$

We now *fix* F_0 and estimate the last term in (35) from above. We may of course assume that $P'(j,G_0,F_0) > 0$ for the fixed triple (j,G_0,F_0) under consideration, as otherwise there is nothing to prove. Thus we may in particular assume that $|F_0|/p_1\binom{n}{2} \ge (2/3)(1+\eta)$. Let us write $\mathcal{B}'(j,G_0,F_0)$ for the set of $\mathbf{G}\in\mathcal{B}'_j$ such that $G^{(j)}_{p_1} = G_0$ and $F^{(j)} = F_0$. To show that $P'(j,G_0,F_0)$ is small, we argue that if $\mathbf{G} = (G^{(i)}_{p_1})_{1\le i\le k} \in \mathcal{B}'(j,G_0,F_0)$, then the edges of the graphs $G^{(i)}_{p_1}$ $(i\ne j,$ $1\le i\le k)$ are distributed in a rather unlikely way.

Let p_2 be such that $1-p_2 = (1-p_1)^{k-1}$, and note that then $p_2/(k-1) \le p_1 = (1+o(1))p_2/(k-1)$. For any given $\mathbf{G} = (G^{(i)}_{p_1})_{1\le i\le k} \in \Omega$, we write $G^{(\neg j)}_{p_2}$ for $\bigcup_i G^{(i)}_{p_1}$, where the union is taken over all $i\ne j$ $(1\le i\le k)$. Thus $G^{(\neg j)}_{p_2}$ is a random element from $\mathcal{G}(n,p_2)$, and $G_p = G^{(j)}_{p_1} \cup G^{(\neg j)}_{p_2}$. Note also that if $\mathbf{G}\in\Omega'$, then $e(G^{(\neg j)}_{p_2}) \le (1+2\varepsilon)p_2\binom{n}{2}$.

For any $\mathbf{G} = (G^{(i)}_{p_1})_{1\le i\le k} \in \Omega$, let $F' = F'(\mathbf{G}) = E(G^{(\neg j)}_{p_2}) \cap E(\Pi_{F_0})$. Also, let $f' = f'(\mathbf{G}) = |F'|$ for any $\mathbf{G}\in\Omega$. Clearly, $f' = f'(\mathbf{G})$ $(\mathbf{G}\in\Omega)$ is binomially distributed with parameters $e(\Pi_{F_0})$ and p_2. In fact, it is clear that $f' = f'(\mathbf{G})$ has this distribution even if we condition on $\mathbf{G}$ being such that $G^{(j)}_{p_1} = G_0$. We now verify the following claim, from which we shall deduce an exponential upper estimate for $P'(j,G_0,F_0)$. In the sequel, we write $\mathbb{E}_{G_0}$ for the expectation in the space $\Omega\cap\{\mathbf{G}\colon G^{(j)}_{p_1} = G_0\}$.

Claim. If $\mathbf{G}\in\mathcal{B}'(j,G_0,F_0)$ *then* $f' \le (1-\eta)\mathbb{E}_{G_0}(f')$.

Proof of the Claim. Let us fix $\mathbf{G} = (G^{(j)}_{p_1})_{1\le j\le k} \in \mathcal{B}'(j,G_0,F_0)$. Let $F^{(\neg j)} = F^{(\neg j)}(\mathbf{G}) = F \cap E(G^{(\neg j)}_{p_2})$, and put $f^{(\neg j)} = f^{(\neg j)}(\mathbf{G}) = |F^{(\neg j)}|$. Clearly $F^{(\neg j)} \cup F' \subset E(G^{(\neg j)}_{p_2})$ and, since F spans no K^4, we have $F^{(\neg j)} \cap F' = \emptyset$. Thus we have $f^{(\neg j)} + f' \le e(G^{(\neg j)}_{p_2}) \le (1+2\varepsilon)p_2\binom{n}{2}$, and hence

$$f' \le (1+2\varepsilon-\gamma^{(\neg j)})p_2\binom{n}{2}, \tag{36}$$

where $\gamma^{(\neg j)}=\gamma^{(\neg j)}(\mathbf{G})=f^{(\neg j)}/p_2\binom{n}{2}$. We now show that $\gamma^{(\neg j)}$ is suitably large. We have $f=|F|\le f^{(j)}+f^{(\neg j)}$, and hence

$$\frac{2}{3}+\eta\le\gamma^{(j)}\frac{p_1\binom{n}{2}}{e(G_p)}+\gamma^{(\neg j)}\frac{p_2\binom{n}{2}}{e(G_p)}\le\frac{1}{k}\left(\frac{1+\varepsilon}{1-\varepsilon}\right)\left(\gamma^{(j)}+(k-1)\gamma^{(\neg j)}\right).$$

Therefore

$$\frac{2}{3}(1+\eta)\le\frac{1}{k}\gamma^{(j)}+\frac{k-1}{k}\gamma^{(\neg j)}\le\frac{1}{k}\gamma^{*}+\frac{k-1}{k}\gamma^{(\neg j)}\le\frac{1}{k}(1+\varepsilon)+\gamma^{(\neg j)},$$

where the last inequality follows from (32). Thus we conclude that $\gamma^{(\neg j)}\ge 2/3$. We now note that $\mu=\mathbb{E}_{G_0}(f')=p_2e(\Pi_{F_0})\ge(2\gamma^*-1-\sigma)p_2\binom{n}{2}$. Note that, in particular, by (31) and the choice of σ, we have $(1/3)p_2\binom{n}{2}\le\mu\le p_2\binom{n}{2}$. Let b denote the right-hand side of (36). Then

$$\begin{aligned}\mu-b&\ge\left(2\gamma^*-1-\sigma-1-2\varepsilon+\gamma^{(\neg j)}\right)p_2\binom{n}{2}\\&\ge\left(\frac{4\eta}{3}-2\varepsilon-\sigma\right)p_2\binom{n}{2}\ge\eta p_2\binom{n}{2}\ge\eta\mu,\end{aligned}$$

and therefore $f'\le b\le(1-\eta)\mu$, as claimed.

We now use our claim to bound $P'(j,G_0,F_0)$, which, we recall, is defined in (34) above. Recall also that $f'\sim\mathrm{Bi}(e(\Pi_{F_0}),p_2)$. We have

$$P'(j,G_0,F_0)\le\mathbb{P}\left\{f'\le(1-\eta)\mathbb{E}_{G_0}(f')\,\middle|\,G_{p_1}^{(j)}=G_0\right\}\le\exp\left\{-\frac{1}{2}\eta^2\mu\right\},$$

where, as remarked before, $\mu=\mathbb{E}_{G_0}(f')=e(\Pi_{F_0})p_2\ge(1/3)p_2\binom{n}{2}$. Thus, from (35) we deduce that

$$\begin{aligned}\mathbb{P}'(\mathcal{B}'_j\mid G_{p_1}^{(j)}=G_0)&\le 2^{(1+\varepsilon)\frac{p}{k}\binom{n}{2}}\exp\left\{-\frac{1}{6}\eta^2\left(1-\frac{1}{k}\right)p\binom{n}{2}\right\}\\&=\exp\left\{\left((1+\varepsilon)(\log 2)-\frac{k-1}{6}\eta^2\right)\frac{p}{k}\binom{n}{2}\right\}\\&\le\exp\left\{-\frac{1}{12}\left(1-\frac{1}{k}\right)\eta^2p\binom{n}{2}\right\}\\&\le\exp\left\{-\frac{1}{30}\eta^2pn^2\right\}.\end{aligned}$$

We now recall (33) to deduce that $\mathbb{P}'(\mathcal{B}'_j)\le\exp\{\eta^2pn^2/30\}=o(1)$, completing the proof of Lemma 18. □

We next show that, loosely speaking, the quantity $\mathrm{ex}(G_{n,p},H)\{p\binom{n}{2}\}^{-1}$ is non-increasing in probability for any fixed graph H. In particular, this shows that Lemma 18 implies Theorem 2.

Lemma 19. *Suppose $0 \leq p = p(n) \leq 1$, $0 < \gamma = \gamma(n) \leq 1$, and $0 < \varepsilon = \varepsilon(n) \leq 1$ are such that $\varepsilon^2 \gamma p n^2 \to \infty$ as $n \to \infty$. Suppose also that $G_{n,p} \to_\gamma H$ holds almost surely for some graph H. Then, if $0 \leq p' = p'(n) \leq 1$ is such that $p' \geq p$ for all large enough n, we almost surely have $G_{n,p'} \to_{\gamma(1+\varepsilon)} H$.*

Proof. Write $\gamma' = \gamma'(n) = \gamma(1+\varepsilon)$. Let $p' = p'(n)$ be as in the statement of our lemma. Suppose for a contradiction that $G_{n,p'} \to_{\gamma'} H$ fails with probability at least $\theta > 0$ for arbitrarily large values of n, where θ is some positive absolute constant. Put $\lambda = \lambda(n) = p(n)/p'(n) \leq 1$. Note that we may generate $G_{n,p}$ by first generating $G_{n,p'}$ and then randomly removing its edges, each with probability $1-\lambda$, and with all these deletions independent. Looking at this method for generating $G_{n,p}$, we shall deduce below that the probability that (*) $G_{n,p} \to_\gamma H$ fails is at least $\theta/3$ for arbitrarily large n, which is a contradiction.

Let $\delta = \varepsilon/4$. For arbitrarily large n, with probability at least $2\theta/3$ we have that (†) $G_{n,p'} \to_{\gamma'} H$ fails and $e(G_{n,p'}) = (1+O_1(\delta))p'\binom{n}{2}$. Suppose that, when generating $G_{n,p}$ by the above method, we first generated a $G_{n,p'}$ satisfying (†) above. Let $J = J(G_{n,p'}) \subset G_{n,p'}$ be an H-free subgraph of $G_{n,p'}$ with $e(J) \geq \gamma' e(G_{n,p'})$. Clearly, the H-free subgraph $J_\lambda = J \cap G_{n,p}$ of J is a subgraph of our $G_{n,p}$. We have $e(J_\lambda) = (1+O_1(\delta))\lambda e(J)$ and $e(G_{n,p}) = (1+O_1(\delta))\lambda e(G_{n,p'})$ with probability $1-o(1)$, and hence we have $e(J_\lambda) \geq \gamma e(G_{n,p})$ with probability $1-o(1)$. Therefore, given that $G_{n,p'}$ satisfies (†) above, the probability that we generate a $G_{n,p}$ for which (*) fails is $1-o(1)$. Since the probability that we generate $G_{n,p}$ satisfying (†) is at least $2\theta/3$ for arbitrarily large n, we conclude that our $G_{n,p}$ will fail to satisfy (*) with probability at least $\theta/3$ for arbitrarily large n, which is the contradiction we were after. □

Proof of Theorem 2. Theorem 2 follows at once from Lemmas 18 and 19. □

A simple variant of the method used in the proof of Lemma 19 gives the following 'equivalence result' between the binomial and the uniform models of random graphs with respect to the property $G \to_\gamma H$.

Lemma 20. *Let H be a graph. Consider the following two assertions.*

$S_{\mathrm{bin}}(\gamma, p)$: *$G_{n,p} \to_\gamma H$ holds almost surely,*
$S_{\mathrm{unif}}(\gamma, M)$: *$G_{n,M} \to_\gamma H$ holds almost surely,*

where $0 < \gamma = \gamma(n) \leq 1$, $0 < p = p(n) \leq 1$, and $0 < M = M(n) \leq \binom{n}{2}$ are arbitrary functions. Suppose $\omega = \omega(n) \to \infty$ as $n \to \infty$. Then the following holds.

(*i*) *Suppose $pn^2 \to \infty$, $\omega = o(np^{1/2})$, and $h = h(n) = \omega n p^{1/2}$. Let $\gamma' = \gamma'(n) = \gamma + 2h/M'$ and $M' = M'(n) = \lceil p\binom{n}{2} + h \rceil$. Then $S_{\mathrm{bin}}(\gamma, p)$ implies $S_{\mathrm{unif}}(\gamma', M')$.*
(*ii*) *Suppose $M = M(n) \to \infty$, $\omega = o(M^{1/2})$, and $h = h(n) = \omega M^{1/2}$. Let $\gamma' = \gamma'(n) = \gamma + 2h/M$, and $p' = p'(n) = (M+h)\binom{n}{2}^{-1}$. Then $S_{\mathrm{unif}}(\gamma, M)$ implies $S_{\mathrm{bin}}(\gamma', p')$.*

Proof. Let us prove (*i*). Assume $S_{\mathrm{bin}}(\gamma, p)$ holds. We may generate $G_{n,M'}$ by first generating $G_{n,p}$ conditioned on the event $E = \{e(G_{n,p}) \leq M'\}$, and then randomly adding $M' - e(G_{n,p})$ edges to it so as to have a graph with M' edges. For clarity, let us write $G^E_{n,p}$ for our binomial random graph $G_{n,p}$ conditioned on E. Note that $\mathbb{P}(E) = 1 - o(1)$, and hence the effect of conditioning on E is, so to speak, negligible. We now claim that $G_{n,M'} \to_{\gamma'} H$ holds with probability $1 - o(1)$.

Suppose our claim fails, and hence, for arbitrarily large n, there exists with probability at least $\theta > 0$ an H-free subgraph $J \subset G_{n,M'}$ of $G_{n,M'}$ with $e(J) \geq \gamma' e(G_{n,M'})$, where θ is some positive absolute constant. Observe that if $J \subset G_{n,M'}$ is an H-free subgraph of $G_{n,M'}$, then, obviously, $J' = J \cap G^E_{n,p} \subset G^E_{n,p}$ is an H-free subgraph of $G^E_{n,p}$. Now note that almost surely we have $M' - e(G^E_{n,p}) \leq 2h$, and hence almost surely $\mathrm{ex}(G^E_{n,p}, H) \geq \mathrm{ex}(G_{n,M'}, H) - 2h$. Since we are assuming that $\mathrm{ex}(G_{n,M'}, H) \geq \gamma' e(G_{n,M'})$ with probability $\theta > 0$ for arbitrarily large n, we have

$$\mathrm{ex}(G^E_{n,p}, H) \geq \gamma' e(G_{n,M'}) - 2h \geq \gamma' M' - 2h \geq \gamma e(G^E_{n,p})$$

with probability $\theta/2$ for arbitrarily large n. Since $G^E_{n,p}$ is the binomial random graph $G_{n,p}$ conditioned on the almost sure event E, we deduce that $\mathrm{ex}(G_{n,p}, H) \geq \gamma e(G_{n,p})$ with probability at least $\theta/3$ for arbitrarily large n, contradicting $S_{\mathrm{bin}}(\gamma, p)$. Thus $S_{\mathrm{unif}}(\gamma', M')$ follows. The proof of (ii) is similar. □

Proof of Corollary 3. Corollary 3 follows easily from Theorem 2 and Lemma 20. □

§5. Deterministic Consequences

In this section we give a few results concerning the existence of very sparse graphs $G = G_\eta$ that satisfy $G \to_{2/3+\eta} K^4$ for any fixed $\eta > 0$.

If H is a graph of order $|H| \geq 3$ and size $e(H) \geq 1$, recall that its 2-density is $d_2(H) = (e(H) - 1)/(|H| - 2)$. For an integer $k \geq 3$, let $\mathcal{H}_k$ be the family of all graphs H with $3 \leq |H| \leq k$, $e(H) \geq 1$, and $d_2(H) > d_2(K^4)$. Also, let $\mathrm{Forb}(\mathcal{H}_k)$ be the collection of all graphs G that are H-free for all $H \in \mathcal{H}_k$. The following result may be proved by the so called 'deletion method'. (For details, see **[8, 9]**.)

Theorem 21. *Let $0 < \eta \leq 1/3$ and $k \geq 3$ be fixed. Then there exists a graph $G = G_{\eta,k} \in \mathrm{Forb}(\mathcal{H}_k)$ such that $G \to_{2/3+\eta} K^4$.* □

We now single out a corollary to Theorem 21. In Corollary 22 below, the property that G belongs to $\mathrm{Forb}(\mathcal{H}_k)$ in Theorem 21 is replaced by a collection of simpler and more concrete conditions. For instance, one of these conditions is that G should not contain a copy of K^5. To state another condition that appears in Corollary 22, we need to introduce a definition.

Let G be a graph. Suppose $K_1, \ldots, K_h$ $(h \geq 2)$ are distinct copies of K^4 in G, and $e_1 \in E(K_1), \ldots, e_{h-1} \in E(K_{h-1})$ are $h - 1$ edges of G such that $E(K_i) \cap \bigcup_{1\leq j<i} E(K_j) = \{e_{i-1}\}$ and $V(K_i) \cap \bigcup_{1\leq j<i} V(K_j) = V(e_i)$ for all $2 \leq i \leq h$. Then we say that $(K_1, \ldots, K_h)$ is an (h, K^4)-*path* in G. Now assume that $(K_1, \ldots, K_h)$ is an (h, K^4)-path in G and that the edge $e \in E(G)$ joins a vertex in $V(K_1) \setminus \bigcup_{1<j\leq h} V(K_j)$ to a vertex in $V(K_h) \setminus \bigcup_{1\leq i<h} V(K_i)$. Then, $(K_1, \ldots, K_h; e)$ is said to be an (h, K^4)-*quasi-cycle* in G.

It is immediate to check that if $(K_1, \ldots, K_h)$ is an (h, K^4)-path, then $H = \bigcup_{1\leq j\leq h} K_j$ has 2-density $d_2(H) = d_2(K^4)$. Also, if $(K_1, \ldots, K_h; e)$ is an (h, K^4)-quasi-cycle, then $H' = H + e$ has 2-density $d_2(H') > d_2(K^4)$.

Corollary 22. *For any $0 < \eta \leq 1/3$ and $k \geq 1$, there is a graph $G = G_{\eta,k}$ such that (i) G contains no K^5, (ii) any two copies of K^4 in G share at most two vertices, (iii) G contains no (h, K^4)-quasi-cycles for any $2 \leq h \leq k$, and (iv) $G \to_{2/3+\eta} K^4$.* □

We remark that Erdős and Nešetřil have raised the question as to whether the graphs $G_{\eta,k}$ as in Corollary 22 exist.

§6. A Conjecture

In this short paragraph we state a conjecture from which, if true, one may deduce Conjecture 1. Let $H = H^h$ be a graph of order $|H| = h \geq 3$ and suppose H has vertices $v_1, \dots, v_h$. Let $0 < p = p(m) \leq 1$ be given. Let also $\mathbf{V} = (V_i)_{i=1}^h$ be a family of h pairwise disjoint sets, each of cardinality m. Suppose reals $0 < \varepsilon \leq 1$ and $0 < \gamma \leq 1$ and an integer T are given. We say that an h-partite graph F with h-partition $V(F) = V_1 \cup \dots \cup V_h$ and size $e(F) = |F| = T$ is an $(\varepsilon, \gamma, H; \mathbf{V}, T)$-*graph* if the pair (V_i, V_j) is (ε, F, p)-regular and has p-density $\gamma \leq d_{F,p}(V_i, V_j) \leq 2$ whenever $v_i v_j \in E(H)$.

Conjecture 23. *Let constants* $0 < \alpha \leq 1$ *and* $0 < \gamma \leq 1$ *be given. Then there are constants* $\varepsilon = \varepsilon(\alpha, \gamma) > 0$ *and* $C = C(\alpha, \gamma)$ *such that, if* $p = p(m) \geq Cm^{-1/d_{23}(H)}$*, the number of* H*-free* $(\varepsilon, \gamma, H; \mathbf{V}, T)$*-graphs is at most*

$$\alpha^T \binom{\binom{h}{2} m^2}{T}$$

for all T *and all sufficiently large* m.

If H above is a forest, Conjecture 23 holds trivially, since, in this case, all $(\varepsilon, \gamma, H; \mathbf{V}, T)$-graphs contain a copy of H. A lemma in Kohayakawa, Łuczak, and Rödl **[12]** may be used to show that Conjecture 23 holds for the case in which $H = K^3$. In fact, this lemma from **[12]** is similar in spirit to Lemma 8 above, although much simpler.

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Instituto de Matemática e Estatística, Universidade de São Paulo, Rua do Matão 1010, 05508–900 São Paulo, SP, Brazil

E-mail address: <yoshi@ime.usp.br>

Department of Discrete Mathematics, Adam Mickiewicz University, ul. Matejki 48/49, 60–769 Poznań, Poland

E-mail address: <tomasz@math.amu.edu.pl>, <tomasz@mathcs.emory.edu>

Department of Mathematics and Computer Science, Emory University, Atlanta, Georgia 30322, USA

E-mail address: <rodl@mathcs.emory.edu>