Mini course : Part 3

"Solenoids, monodromy and horseshoes"

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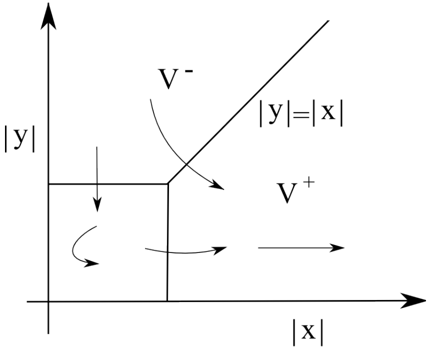
(Floripadynsys 2013)

Complex Hénon mappings in $\mathbb{C}^2$

- **Basic properties:** the map $\binom{x}{y} \mapsto \binom{x^2+c-ay}{x}$ has inverse $\binom{x}{y} \mapsto \binom{y}{(1/a)(y^2+c-x)}$ and constant jacobian equal to a.
- **Crude picture of dynamics:**



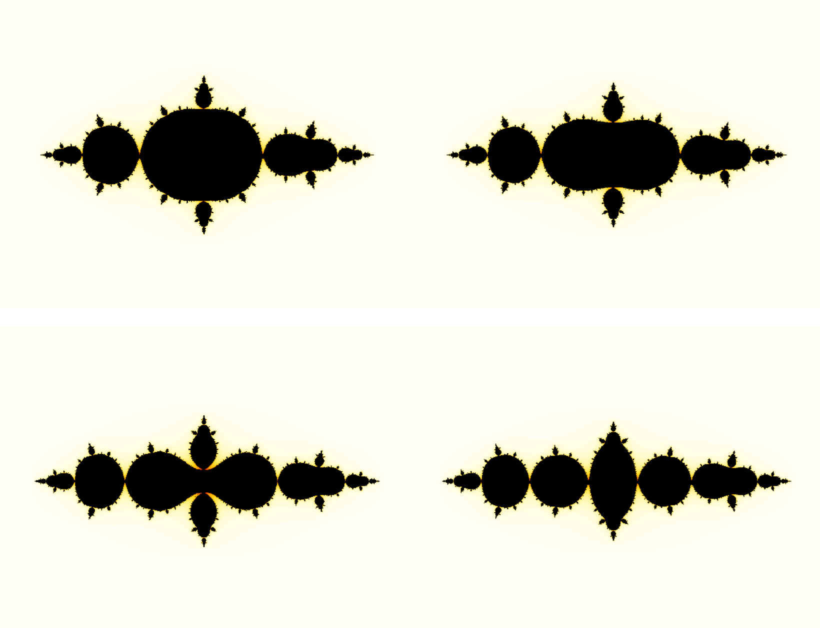
Basic invariant sets

- **Filled Julia sets**

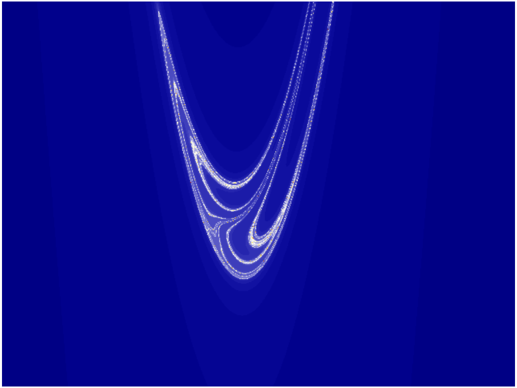
$$K^+ := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{C}^2 \text{ with bounded forward orbit} \right\}$$

- **Escaping set:** $U^+ := \mathbb{C}^2 - K^+$. We have also $U^+ = \bigcup_{n \geq 0} H^{-n}(V^+)$.
- **Julia sets** $J^+ = \partial K^+$, also $J^- = \partial K^-$. We can define $K = K^+ \cap K^-$.
- **Basins of attraction** $W^s(p)$ **of fixed points**.
- **Stable and unstable manifolds of saddle points** p: they are isomorphic to $\mathbb{C}$.
- What is the topology of these sets? Are they connected?
- (partial) answer: $K^{\pm}$ is always connected.

Examples of horizontal slices of non-escaping sets



Picture Stable manifold, real slice



Self-similarity in the unstable manifolds

- **The parametrization:** Pictures of the stable manifolds look self-simila, here is why:

Theorem (Hubbard)

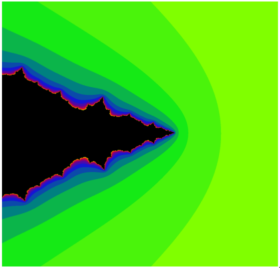
Let p be a saddle fixed point for H and assume that the eigenvalues of $DH(p)$ are $|\lambda| > 1 > |\mu|$. Let v be an eigenvector for λ. The limit

$$\Phi(z) = \lim_{m \to \infty} H^m \left(p + \frac{z}{\lambda^m v} \right)$$

exists and give an injective immersion of $\mathbb{C}$ onto the unstable manifold $W^u(p)$ that satisfies

$$\Phi(\lambda z) = H(\Phi(z))$$

Examples of unstable manifolds of saddle points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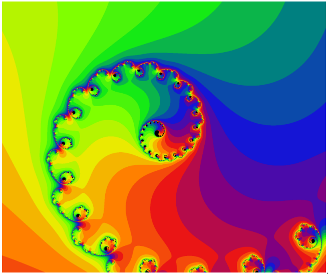
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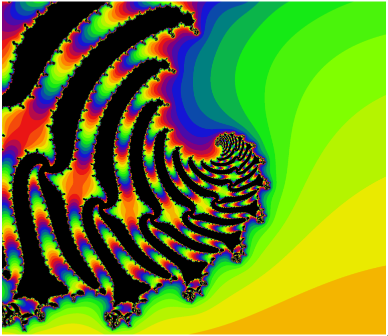
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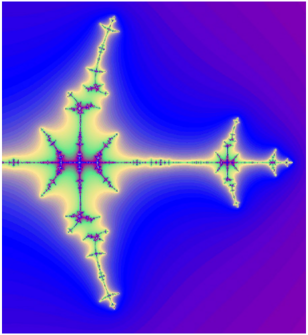


Focussing on the escaping sets

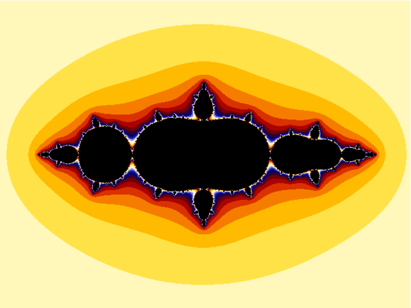
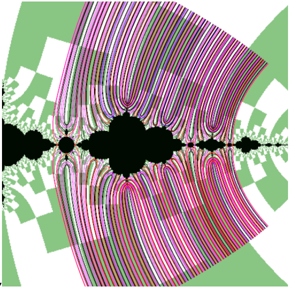
- **Equivalent of a Böttcher coordinate:**

Theorem (Hubbard)

There exists an analytic map $\phi^+ : V^+ \to \mathbb{C}$ satisfying the functional equation:

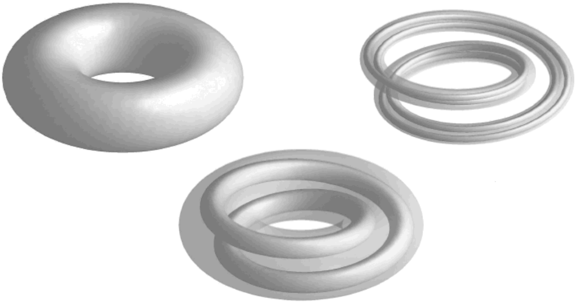
$$\phi^+ \circ H = (\phi^+)^2$$

- **External rays and equipotentials:** (second pic made by R.Oliva)

,


The analogue of the disk to be pinched

- **Solenoidal maps**: self-map of solid torus $T_0 = S^1 \times \mathbb{D}$ given by $\sigma : (\zeta, z) \mapsto \left(\zeta^2, \frac{1}{2}\zeta + \epsilon\frac{z}{\zeta}\right)$.



- **Key fact:** σ can be extended to the 3-sphere $S^3 = T_0 \cup T_1$.
- **Invariant solenoids:** we define $\Sigma^{\pm}$

$$\Sigma^+ := \bigcap_{n \geq 0} \sigma^n(T_0)$$

- **Cone over solenoid:** Cone(Σ^- as $\left\{(r,\theta) \in \mathbb{R}^4 | r \geq 1, \theta \in \Sigma^-\right\}$.

Topological model for $c \in \mathcal{C}$ and a small

- Let us set $Y = \mathbb{R}^4 - Cone(\Sigma^-)$. And consider the map $g : Y \to Y$ defined by:

$$g(r, \theta) = (r^2, \sigma(\theta)).$$

- Let $H_{a,c} : \binom{x}{y} \to \binom{x^2+c-ay}{x}$.

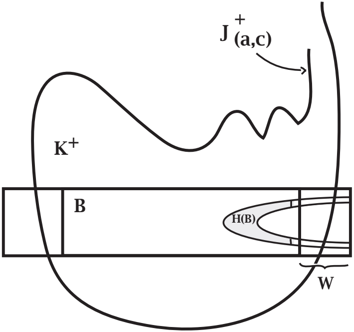
Theorem (B.)

For any $c \in \mathcal{C}$, there exists $\epsilon > 0$ such that: for all $0 < |a| < \epsilon$ there is a homeomorphism

$$h : \mathbb{C}^2 \to Y$$

which conjugates $H_{a,c}$ to g.

Picture of model



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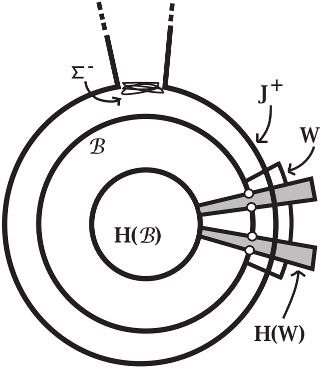
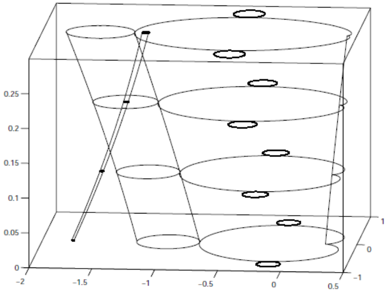
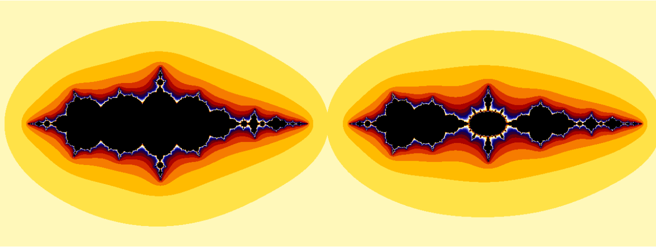


What we have done so far

- **The analogue of the closed disk, to be pinched:** this analogue is the set K^+ (topologically, a closed 4-ball, minus a solenoid in the boundary 3-sphere).
- **Where to pinch?** In the filled Julia set K^+, the pinching occur along stable manifolds of saddle cycles. Observe that those are dense in J^+.

"Truly 2-dimensional" examples





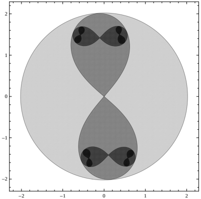
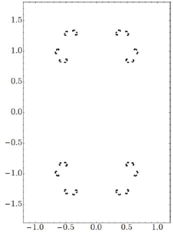
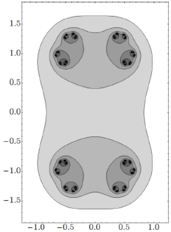
What could be done

Claim (Model for J^+)

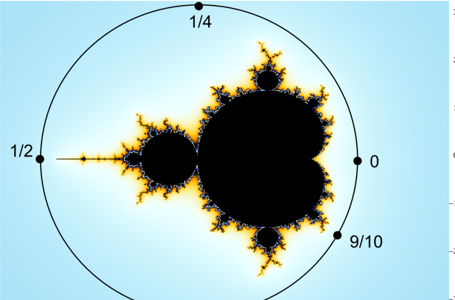
In the solid torus T, pick an identification between the disks $D_{3/7}$ and $D_{4/7}$ that preserves the dynamics. In $S^3 - \Sigma^-$, consider the following equivalence relation : $x \sim y$ if and only if there exists $N \in \mathbb{N}$ such that $\sigma_0^{\circ N}(x)$ belongs to one of the disks $D_{3/7}, D_{4/7}$, $\sigma_0^{\circ N}(y)$ belongs to the other disk and both points are identified in these disks. The space $(S^3 - \Sigma^-)/\sim$ together with the solenoidal map induced on it is a model for J^+ together with the induced action of the Hénon map H.

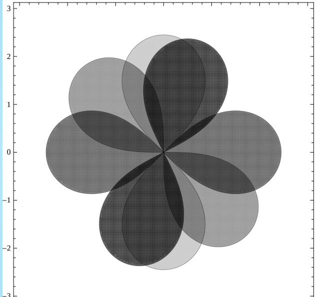
One dimensional dynamics and automorphisms of the shift

- **Fact:** $c \notin \mathcal{M} \Rightarrow K_c$ is a Cantor set. **Idea of the proof:**

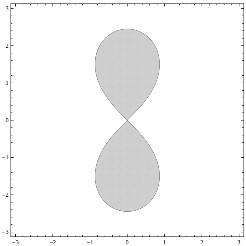
- **Circling around $\mathcal{M}$:** makes the Cantor sets dance...





Shift and quadratic dynamics

- **coding the orbit of $z \in J_c = K_c$:** Let D be a large disk with the critical value c on the boundary circle. Label D_0 and D_1 the 2 components of $P_c^{-1}(D)$.



- **Itinerary of $z \in J_c$:** $I(z) := (s_0, s_1, \dots)$ where $s_j = 0$ if $P_c^j(z) \in D_0$ (and 1 otherwise).

Proposition

The "itinerary map" $I : J_c \to \Sigma_2$ is a homeomorphism and the restriction of P_c to J_c is conjugated to the shift: $\sigma \circ I = I \circ P_c$

Shift and quadratic dynamics II

Theorem (Blanchard-Devaney-Keen)

Suppose θ is the automorphism of the 2-shift induced by the monodromy associated to a closed curve which winds once around $\mathcal{M}$ in θ interchanges 0 's and 1's in Σ_2.

- **"Shift locus":** deg d polynomials with all crit. points going to ∞

Theorem

If $f \in \mathcal{S}_d$ then $K(f)$ is homeomorphic to a Cantor set, and $f_{|K(f)}$ is topologically conjugate to the one-sided shift map on d symbols.

Theorem (Blanchard-Devaney-Keen)

The homomorphism

$$\pi_1(\mathcal{S}_d, f_0) \to Aut(\sigma, \Sigma_d)$$

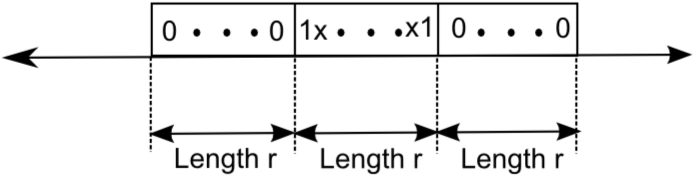
is surjective in every degree $d \geq 2$.

Automorphisms of the full 2-shift

- $Aut(\Sigma_2^{\pm})$ **is huge:** it contains a countable sum of copies of $\mathbb{Z}$, and even better:

Proposition

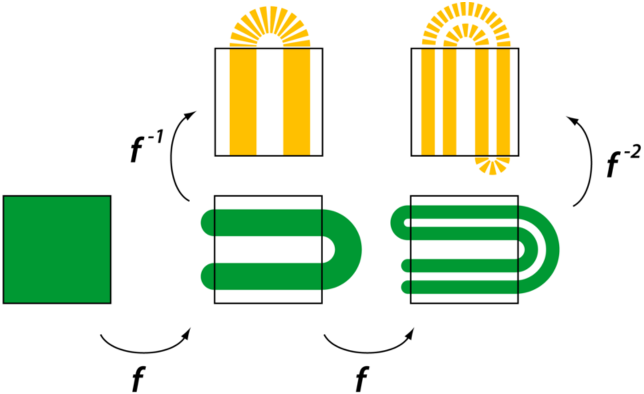
$Aut(\Sigma_2^{\pm})$ contains every finite group.



- **Proof**: Cayley's theorem $\Rightarrow$ it is enough to show $S_n \subset Aut(\Sigma_2^{\pm})$. Given n, find r large enough so that there exists at least n distinct blocks $B_1, \ldots B_n$ of length r starting and finishing with 1. Define an automorphism ϕ by: whenever it sees one block with a left and right padding of 0s, it permutes the B_n.
- **Open questions :** is $Aut(\Sigma_2^{\pm}) \simeq Aut(\Sigma_3^{\pm})$?

Hénon maps: horseshoes and shift locus

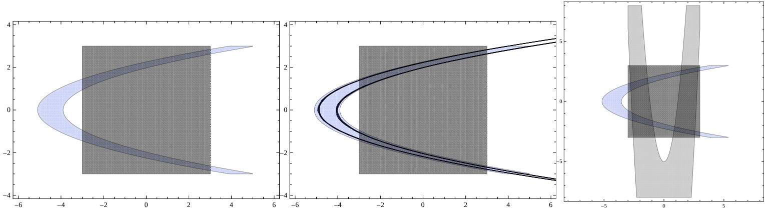
- **Real horseshoe in $\mathbb{R}^2$:**



Proposition

$K^+ = B \cap f^{-1}(B) \cap f^{-2}(B) \dots$ is Cantor set $\times [0,1]$, so is K^- and $K = K \cap K^-$ is a Cantor set.

- **(Real) horseshoe realized by a Hénon map**: take $a \in \mathbb{R}$ small, $c \in \mathbb{R}$ very negative:

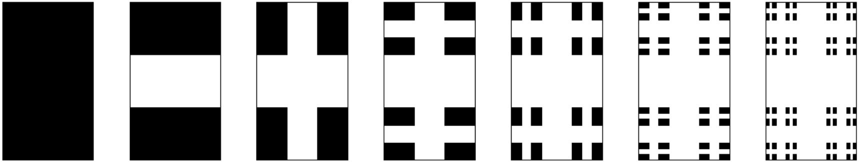


Definition

The complex horseshoe locus $\mathcal{H}^{\mathbb{C}}$ is the set of parameters $(a, c) \in \mathbb{C}^2$ for which the restriction of $H_{a,c}$ is hyperbolic and topologically conjugate to the full 2-shift.

Horseshoe and symbolic dynamics

- **Real horseshoe in $\mathbb{R}^2$:**

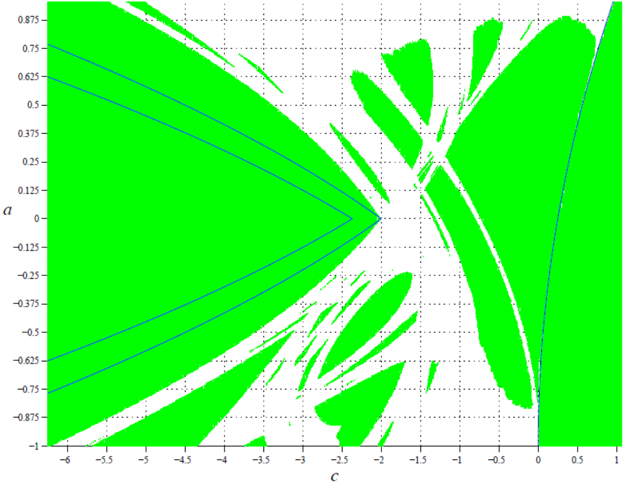


Proposition (Hubbard-ObersteVorth)

A Hénon map satisfies the horseshoe condition for some $R > 0$ if

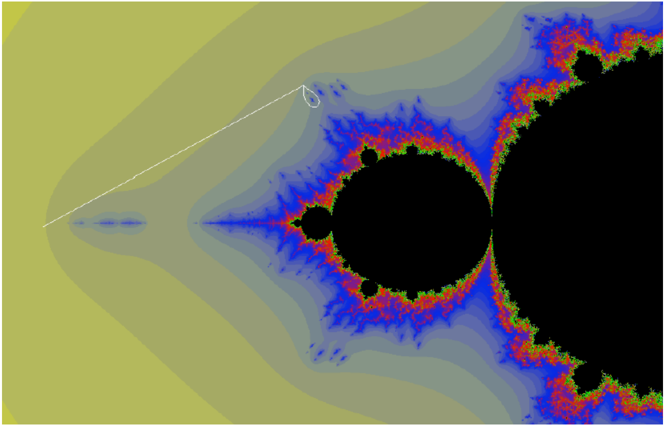
$$|c| > 2(1 + |a|)^2).$$

- **Hyperbolicity locus**: it contains a lot more than just the "shift locus":



Examples of monodromies: work of Hubbard and Lipa

- **Example**: take a slice $a = 0.3$ in the parameter space:



Claim: this loop should correspond to the automorphism
$AAB \star BABAA$.

Monodromy: conjectures

Open question

Show that the loops drawn actually stay entirely within the Horseshoe locus.

- **The main conjecture:** obtain the analogue of the Blanchard-Devaney-Keen result:

Conjecture (Hubbard)

The induced monodromy group of the horseshoe locus together with the shift generate $Aut(\Sigma_2^{\pm})$.