Chapter 9
A Strange Attractor

The perfect square has no corners.
Great talents ripen late.
The highest notes are hard to hear.
The greatest form has no shape.

Lao Tzu

9.1 Introduction

Using the Hartman–Grobman theorem, we can show that a small perturbation f of a hyperbolic toral automorphism is conjugate to this automorphism. For such a transformation, all points are therefore nonwandering, and there exists a dense set of recurrent points.

Consider the hyperbolic automorphism on $\mathbf{T}^2$ given by the matrix $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. What happens if we carry out a local perturbation in the neighborhood of the origin $(0,0)$ that transforms this fixed point into an attracting point? By the Hartman–Grobman theorem, there exists an open set U of points that will be attracted by $(0,0)$. In this chapter, we consider an explicit example of perturbation. The open set U in this example is depicted in Fig. 9.1. What can be said about this set?

We will see that there exists a hyperbolic fixed point p on the boundary of this open set. Its stable and unstable manifolds

$$\begin{aligned} W^{\mathrm{ss}}(p) &= \{x \in \mathbf{T}^2 \mid d\big(f^n(x), f^n(p)\big) \xrightarrow[n\to\infty]{} 0\} \\ &= \{x \mid d\big(f^n(x), p\big) \xrightarrow[n\to\infty]{} 0\}, \end{aligned}$$

$$\begin{aligned} W^{\mathrm{su}}(p) &= \{x \in \mathbf{T}^2 \mid d\big(f^{-n}(x), f^{-n}(p)\big) \xrightarrow[n\to\infty]{} 0\} \\ &= \{x \mid d\big(f^{-n}(x), p\big) \xrightarrow[n\to\infty]{} 0\} \end{aligned}$$

form two immersed submanifolds of dimension 1. The stable manifold of p cannot, of course, belong to the open set U of points attracted by $(0,0)$. We will show that its closure K has empty interior and coincides with the complement of U. Consequently, most orbits converge to the origin.

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What can we be said about the dynamics of f^{-1}? The origin is now a repelling point, and all other points converge to the invariant compact set K. Moreover, the transformation restricted to K is transitive. The structure of K is interesting: it has empty interior in $\mathbf{T}^2$, but contains an immersed submanifold of dimension 1 that is both dense in K and has empty interior in K. This is therefore a geometric object that is halfway between a line and a plane. It is indicated in black on Fig. 9.1, while its complement, in white, corresponds to the open set U.

The proofs are based on the Hartman–Grobman linearization theorem and on the existence of an invariant direction on K that is dilated by the differential of f. In fact, the transformation A has an eigenvalue that is greater than 1 and the associated dilation is undisturbed by the perturbation when we are far from the origin.

Historically, the compact set K is the first example of a uniformly hyperbolic attractor that is not a submanifold. It was constructed by S. Smale in 1972. Since the transformation f^{-1} comes from a toral automorphism, which is the simplest example of an Anosov diffeomorphism, it is called a *diffeomorphism derived from Anosov* (DA diffeomorphism for short). We can carry out this type of construction on any transformation with a hyperbolic fixed point.

9.2 Perturbation of a Toral Automorphism

We begin with the matrix $A = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$. We denote the golden mean by $\lambda = \frac{1+\sqrt{5}}{2} \simeq 1.618$. The matrix A admits two eigenvalues λ^2 and λ^{-2}; the associated eigenvectors $\mathbf{e}_{\mathrm{u}} = \frac{1}{\sqrt{1+\lambda^2}}\left(\begin{smallmatrix} \lambda \\ 1 \end{smallmatrix}\right)$ and $\mathbf{e}_{\mathrm{s}} = \frac{1}{\sqrt{1+\lambda^2}}\left(\begin{smallmatrix} -1 \\ \lambda \end{smallmatrix}\right)$ form an orthonormal basis for $\mathbf{R}^2$. We have

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = \frac{1}{\sqrt{1+\lambda^2}} \begin{pmatrix} \lambda & -1 \\ 1 & \lambda \end{pmatrix} \begin{pmatrix} \lambda^2 & 0 \\ 0 & \lambda^{-2} \end{pmatrix} \frac{1}{\sqrt{1+\lambda^2}} \begin{pmatrix} \lambda & 1 \\ -1 & \lambda \end{pmatrix}.$$

Let us perturb A in such a manner that the point 0 becomes attracting. For $(x, y) \in \left[-\frac{1}{2}, \frac{1}{2}\right]^2$, set

$$\begin{aligned} f\begin{pmatrix} x \\ y \end{pmatrix} &= \frac{1}{\sqrt{1+\lambda^2}} \begin{pmatrix} \lambda & -1 \\ 1 & \lambda \end{pmatrix} \begin{pmatrix} \lambda^2 + p_1 k(r/a) & 0 \\ 0 & \lambda^{-2} \end{pmatrix} \frac{1}{\sqrt{1+\lambda^2}} \begin{pmatrix} \lambda & 1 \\ -1 & \lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \\ &= \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \frac{p_1}{1+\lambda^2} k(r/a) \begin{pmatrix} \lambda^2 & \lambda \\ \lambda & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \end{aligned}$$

with $r = \sqrt{x^2 + y^2}$ and $k(r) = (1 - r^2)^2 \mathbf{1}_{[-1,1]}(r)$ used as a C^1 "bump". The parameter a controls the extent of the perturbation, while the parameter p_1 controls its amplitude. When $a \in [0, 1/2]$, the map f passes to the quotient and defines a transformation from $\mathbf{T}^2$ to $\mathbf{T}^2$, also denoted by f. Let us establish some of its properties.

Properties

- *For every* $(x, y) \in \mathbf{T}^2$*, we have* $f((x, y) + \mathbf{R}\mathbf{e}_{\mathrm{u}}) \subset f(x, y) + \mathbf{R}\mathbf{e}_{\mathrm{u}}$.
- *For* $p_1 \in (-\lambda^2, 0]$ *and* $a \in [0, 1/2]$*, the map* f *is a diffeomorphism of the torus* $\mathbf{T}^2$.
- *For* $p_1 \in (-\lambda^2, 1 - \lambda^2]$*, the point* 0 *is an attracting fixed point. We denote its basin of attraction by* U.
- *For* $p_1 \in (-\lambda^2, 1 - \lambda^2]$*, the map* f *has a fixed point* $p \in (0, a)\mathbf{e}_{\mathrm{u}}$ *such that* $[0, p) \subset U$.
- *The open ball* $B(0, |p|)$ *is included in the basin of attraction* U *of* 0.
- *For every* $(x, y) \in U^c$*, we have* $|d_{(x,y)}f \cdot \mathbf{e}_{\mathrm{u}}| > 1$.

Proof

- For every $(x, y) \in \mathbf{T}^2$, the point $f(x, y) - A(x, y)$ belongs to $\mathbf{R}\mathbf{e}_{\mathrm{u}}$. Consequently, the point $f\big((x, y) + t\mathbf{e}_{\mathrm{u}}\big) - A\big((x, y) + t\mathbf{e}_{\mathrm{u}})\big)$ is also in $\mathbf{R}\mathbf{e}_{\mathrm{u}}$, and therefore

$$f\big((x, y) + t\mathbf{e}_{\mathrm{u}}\big) - f(x, y) \in \mathbf{R}\mathbf{e}_{\mathrm{u}}.$$

- Let us determine the Jacobian of f in the orthonormal basis $(\mathbf{e}_{\mathrm{u}}, \mathbf{e}_{\mathrm{s}})$:

$$\begin{aligned} \det(df) &= \frac{\partial}{\partial x}\big(x + \lambda^{-2} p_1 x k(r/a)\big) \\ &= 1 + \lambda^{-2} p_1 k(r/a) + \lambda^{-2} p_1 \frac{x^2}{ra} k'(r/a) \\ &\geqslant 1 + \lambda^{-2} p_1. \end{aligned}$$

  The map f is therefore a local diffeomorphism.

  Let us show that it is bijective. Let S_r be the circle with radius r and center 0. The transformation f restricted to S_r is linear, and $f(S_r)$ is an ellipse with minor axis $\lambda^{-2}\, r\, \mathbf{e}_{\mathrm{s}}$ and major axis $(\lambda^2 + p_1 k(r/a))\, r\, \mathbf{e}_{\mathrm{u}}$. The lengths of these two axes are strictly increasing functions for $r \in [0, a]$. The sets $f(S_r)$ for $r \in [0, a]$ are therefore disjoint, and the transformation f is bijective from the ball $B(0, a)$ onto the set $f(B(0, a))$. This set coincides with the interior of the ellipse $f(S_a)$; it therefore equals the image of $B(0, a)$ by the matrix $\left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$. Outside of $B(0, a)$, this matrix coincides with f, which is bijective.
- The fixed point 0 is attracting for the values given above because the differential $D_0 f$ admits λ^{-2} and $\lambda^2 + p_1$ as eigenvalues.
- The map $h(t) = \lambda^2 t + p_1 t\, k(t/a)$ admits a fixed point in the interval $(0, a)$ because $h(0) = 0$, $h'(0) \in (0, 1)$, and $h(a) > a$. Let t_0 be the smallest fixed point of h in $(0, a)$. We set $p = t_0\, \mathbf{e}_{\mathrm{u}}$ and note that $\lambda^2 + p_1 k(|p|/a) = 1$.
- Let us show that $|f(x, y)| < |(x, y)|$ if $|(x, y)| < |p|$. In the basis $(\mathbf{e}_{\mathrm{u}}, \mathbf{e}_{\mathrm{s}})$, we have

$$\begin{aligned} |f(x, y)|^2 &= \lambda^{-4} y^2 + \big(\lambda^2 x + p_1 x k(r/a)\big)^2 \\ &< \lambda^{-4} y^2 + \big(\lambda^2 x + p_1 x k(|p|/a)\big)^2 \\ &= \lambda^{-4} y^2 + x^2. \end{aligned}$$

We have used the fact that k is strictly decreasing on $(0, a)$ and the equality $p_1 k(|p|/a) = 1 - \lambda^2$. The function $(x, y) \mapsto f(x, y)/|(x, y)|$ reaches its maximum on the annulus $\{\varepsilon \leqslant |m| \leqslant |p| - \varepsilon\}$. It is therefore contractive on this annulus. Every point of $B(0, |p|)$ has an orbit that ends up entering $B(0, \varepsilon)$. The orbit of the point therefore converges to 0.

- Let us determine the differential in the direction $\mathbf{e}_\mathrm{u}$. In the basis $(\mathbf{e}_\mathrm{u}, \mathbf{e}_\mathrm{s})$, we have

$$\begin{aligned} d_{(x,y)}f.\mathbf{e}_\mathrm{u} &= \lambda^2 + p_1 k(r/a) + p_1 \frac{x^2}{ra} k'(r/a) \\ &= 1 - p_1\big(k(|p|/a) - k(r/a)\big) + p_1 \frac{x^2}{ra} k'(r/a). \end{aligned}$$

This is greater than 1 if $r \geqslant |p|$ and equal to 1 if $(x, y) = (0, |p|)$. This point is in U.

□

From here on, we take $p_1 = -2.236$ and $a = 0.5$. We denote the basin of attraction of 0 by U and the complement of U by K. Finally, we fix a linearization φ from a neighborhood V of p to $(0, 1)^2$.

9.3 Perturbed Dynamics

We wish to show that the map f restricted to K is transitive. For the proof, we study the stable and unstable manifolds of the fixed point p on the boundary of U. We will need to verify that $W^{\mathrm{su}}(p)$ is dense in $\mathbf{T}^2$, and then that $W^{\mathrm{ss}}(p)$ is dense in K.

Lemma 9.1 *Let $x \in K$. Then for every $\varepsilon > 0$, the segment $x - [0, \varepsilon]\,\mathbf{e}_\mathrm{u}$ meets U. The open set U is therefore dense in $\mathbf{T}^2$, and K has empty interior.*

Proof The open set U consists of the points whose iterates converge to 0; it is invariant under f. If the segment $x - [0, \varepsilon]\,\mathbf{e}_\mathrm{u}$ does not meet U, then the same holds for all of its iterates. Since $D_y f \cdot \mathbf{e}_\mathrm{u} > 1$ if $y \in K$, these iterates are of the form $f^n(x) - [0, c_n \varepsilon]\,\mathbf{e}_\mathrm{u}$, with $c_n \geqslant C^n$ for some constant $C > 1$.

Since the set $\mathbf{R}_+ \mathbf{e}_\mathrm{u}$ is dense in $\mathbf{T}^2$, we can find $n \in \mathbf{N}$ such that every point of $\mathbf{T}^2$ is at a distance less than $|p|$ from $[0, C^n \varepsilon]\,\mathbf{e}_\mathrm{u}$. In particular, the point $f^n(x)$ is at a distance less than $|p|$ from $[0, C^n \varepsilon]\,\mathbf{e}_\mathrm{u}$; in other words, the point 0 is at a distance less than $|p|$ from the subset $f^n(x) - [0, C^n \varepsilon]\,\mathbf{e}_\mathrm{u} \subset K$. This gives a contradiction. □

Proposition 9.1 *The set $p + \mathbf{R}_+ \mathbf{e}_\mathrm{u}$ is included in $W^{\mathrm{su}}(p)$. The unstable manifold $W^{\mathrm{su}}(p)$ is therefore dense in $\mathbf{T}^2$.*

Proof Suppose, to the contrary, that the set is not included in $W^{\mathrm{su}}(p)$. We can then set

$$t_1 = \inf\{t \in \mathbf{R}_+ \mid p + t\mathbf{e}_\mathrm{u} \notin W^{\mathrm{su}}(p)\}.$$

Because of the form of the map $t \mapsto f(p + t\mathbf{e}_u)$, the real number t_1 is positive. The image of $p + [0, t_1)\,\mathbf{e}_u$ by f is of the form $p + [0, s)\,\mathbf{e}_u$. Since $W^{su}(p)$ is invariant under f, we have $s = t_1$, and the point $p' = p + t_1\,\mathbf{e}_u$ is a fixed point of f.

The point p' is distinct from the origin. Indeed, since p is in the set $\mathbf{R}_+\mathbf{e}_u$, we would otherwise have $0 \in \mathbf{R}^*\mathbf{e}_u$, which contradicts the irrationality of λ. The set U is the basin of attraction of the origin. It follows that the fixed point p' is not in U; the slope of the curve $t \mapsto f(p + t\mathbf{e}_u)$ at t_1 is therefore greater than 1. Consequently, the points on $p + [0, t_1)\,\mathbf{e}_u$ close to $p' = p + t_1\mathbf{e}_u$ have negative iterates that approach both p and p', which is absurd. ☐

Proposition 9.2 *Let $m \in U$ and $t > 0$ be such that $m + [0, t)\,\mathbf{e}_u \subset U$ and $m + t\mathbf{e}_u \notin U$. Then $m + t\mathbf{e}_u$ belongs to $W^{ss}(p)$. Moreover, the set $W^{ss}(p) \cap W^{su}(p)$ is dense in K.*

Proof Let $\varphi : V \to (-1, 1)^2$ be a linearization on an open neighborhood of p. Since $W^{su}(p) \cap U$ contains $(0, p)$, there exists $x' \in (0, p) \cap V$ such that $[x', f(x')]$ is in $U \cap V$. Hence, there exists in $\varphi(V)$ a rectangle $[-\delta, \delta] \times [x', f(x')]$ contained in $\varphi(U)$. Its positive iterates under the action of D_pf^{-1} are also in $\varphi(U)$ and cover $[-\delta, \delta] \times [x', p)$. This reasoning is illustrated by Fig. 9.2. The open set U comes to lean against the stable manifold of p.

Consider a curve in the open set $[-\delta, \delta] \times [x', -x')$ originating in the lower half-plane, and not entirely contained in $\varphi(U)$. The first point of the curve that is not in $\varphi(U)$ must lie on the x-axis, that is, on $\varphi(W^{ss}(p))$.

For large n, the iterate $f^n(m + [0, t]\,\mathbf{e}_u)$ is a line from a small neighborhood of 0 in the direction of $\mathbf{e}_u$. The first point of the curve that belongs to K must therefore be in V, and belongs to $W^{ss}(p)$.

Let us now show the density of $W^{ss}(p) \cap W^{su}(p)$ in K. Let $m' \in K$ and $\varepsilon > 0$ be such that $m = m' - \varepsilon\,\mathbf{e}_u$ is in U. Since $p + \mathbf{R}_+\mathbf{e}_u$ is dense in $\mathbf{T}^2$, there exists $C > 0$ such that $p + C\,\mathbf{e}_u$ is arbitrarily close to m. Taking up the previous reasoning, we see that the iterate $f^n(p + [C, C + 2t]\,\mathbf{e}_u)$ is close to $f^n(m + [0, 2t]\,\mathbf{e}_u)$. It therefore meets $W^{ss}(p)$ at a point x such that $f^{-n}(x)$ is as close to $m + t\,\mathbf{e}_u$ as we want. ☐

9.4 Transitivity and the Mixing Property

Corollary 9.1 *The map f restricted to K is transitive and topologically mixing.*

Proof Let U_1 be an open set intersecting K, let $x_1 \in W^{ss}(p) \cap W^{su}(p) \cap U_1$, and let n_1 be such that $f^{-n}(x_1)$ is in V for every $n \geqslant n_1$. Set $x_1' = f^{-n_1}(x_1)$. We begin by showing that every segment in the direction of $\mathbf{e}_u$, passing close to x_1', meets $W^{ss}(x_1')$ in the neighborhood of the point x_1'.

Let $R \subset V$ be a small rectangle with center x_1' and oriented in the directions of $\mathbf{e}_s$ and $\mathbf{e}_u$. There exists N such that for every $n \geqslant N$, the iterate $f^n(x_1')$ is in V. The set $f^N(R)$ contains a small rectangle R' with center $f^N(x_1')$ and, after increasing N if necessary, we may assume that R' crosses the open set V from top to bottom.

Since $f^N(x_1')$ is in $\varphi((-1,1)\times\{0\})$, the vertical lines in R' meet $\varphi((-1,1)\times\{0\}) \subset W^{\mathrm{ss}}\big(f^N(x_1')\big)$ in the neighborhood of $f^N(x_1')$. The vertical lines of R therefore meet $W^{\mathrm{ss}}(x_1')$ in the desired manner. Figure 9.3 summarizes the situation.

Let U_2 be another open set intersecting K. To prove the transitivity, it suffices to construct a point $x' \in K$ with a negative iterate in U_2 and a positive iterate in U_1. Let $x_2 \in U_2 \cap W^{\mathrm{ss}}(p)$ and $\varepsilon > 0$ be such that $x_2 + [-\varepsilon, \varepsilon]\,\mathbf{e}_{\mathrm{u}}$ is in U_2. For large n, the image $f^n(x_2 + [-\varepsilon, \varepsilon]\,\mathbf{e}_{\mathrm{u}})$ is a segment in the direction of $\mathbf{e}_{\mathrm{u}}$, close to p, which crosses V from top to bottom. It therefore meets $W^{\mathrm{ss}}(x_1') \cap f^{n_1}(U_1)$ at a point x' that is in K.

The previous reasoning shows that for every sufficiently large n, the set $f^n(U_2) \cap K$ meets $f^{n_1}(U_1)$. This implies that the restriction of f to K is topologically mixing. $\square$

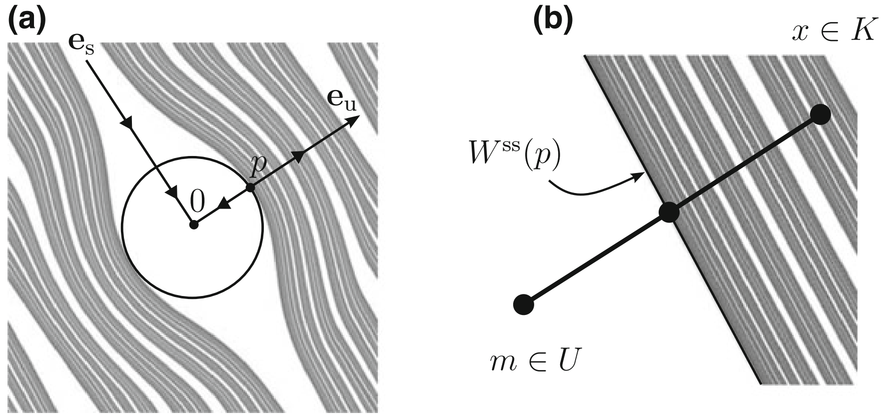

Fig. 9.1 Basin of attraction of the origin. (**a**) 0 is an attracting fixed point. p is a hyperbolic fixed point. (**b**) A segment in the direction $\mathbf{e}_{\mathrm{u}}$ that joins U to K must meet the stable manifold of p

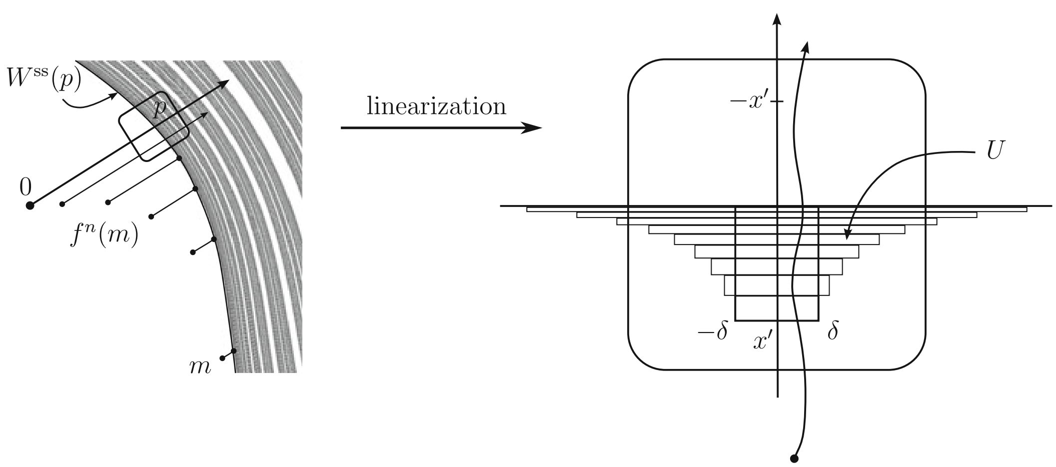

Fig. 9.2 The stable manifold of p

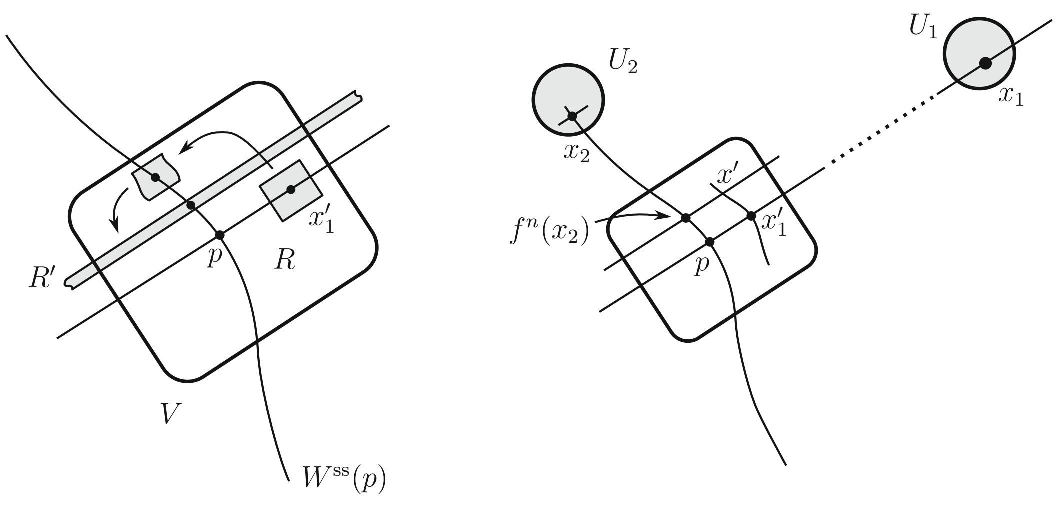

Fig. 9.3 Proof of the transitivity

9.5 Exercises

9.5.1 Basic Exercises

Exercises 1–5 and 8–11 concern the diffeomorphism f we have just studied.

Exercise 1 Show that K is compact, connected, and uncountable.

Exercise 2 Show that for every $m \in \mathbf{T}^2$, we have $f(-m) = -f(m)$. Deduce that there exists a hyperbolic fixed point $p' \in [-a, 0]\,\mathbf{e}_\mathrm{u}$ and that $W^{\mathrm{ss}}(p')$ is dense in K.

Exercise 3 Show that $W^{\mathrm{ss}}(p)$ has empty interior in K.
Hint: Note that no point of $W^{\mathrm{ss}}(p)$ has dense orbit.

Exercise 4 Let $\varepsilon > 0$. Show that $(p+[0,\varepsilon]\,\mathbf{e}_\mathrm{u}) \cap K$ is compact, without any isolated points, and with empty interior in $p + [0,\varepsilon]\,\mathbf{e}_\mathrm{u}$. Deduce that it is uncountable.

Exercise 5 Show that the points of $W^{\mathrm{ss}}(p)$ can be reached from U in the following sense: for every $x \in W^{\mathrm{ss}}(p)$ there exists a continuous map $\gamma : [0,1] \to \mathbf{T}^2$ such that $\gamma([0,1)) \subset U$ and $\gamma(1) = x$.

Exercise 6 Show that we can glue two systems derived from Anosov in such a way that we obtain a diffeomorphism f on a surface of genus 2 whose nonwandering set is the union of two uncountable connected compact sets K_1 and K_2 restricted to which f is transitive.

Exercise 7 Let M be a differential manifold, and let $f : M \to M$ be a C^1 diffeomorphism with a hyperbolic fixed point p. Show that if $W^{\mathrm{ss}}(p)$ and $W^{\mathrm{su}}(p)$ are dense in M, then f is topologically mixing.

9.5.2 More Advanced Exercises

Exercise 8 Show that K is not locally connected.
Hint: Note that every neighborhood of p contains a point of U that belongs to $W^{\mathrm{su}}(p)$ and iterate a neighborhood of this point.

Exercise 9 Let $\gamma : [0,1] \to K$ be a continuous map starting at p: $\gamma(0) = p$. On an open neighborhood V of p on which we have a linearization, we consider a partial path $\gamma([0,\delta])$ contained in V. Show that $\gamma([0,\delta]) \subset W^{\mathrm{ss}}(p)$.

Does it follow that $\gamma([0,1]) \subset W^{\mathrm{ss}}(p)$?

Exercise 10 Let $\gamma : [0,1] \to \mathbf{T}^2$ be a continuous map that satisfies $\gamma([0,1)) \subset W^{\mathrm{ss}}(p)$. Show that $\gamma(1) \in W^{\mathrm{ss}}(p)$.
Hint: Use contradiction and show that $\gamma(1)$ is a hyperbolic fixed point.

Exercise 11 Show that K is not path-connected.
Hint: Show that p and $-p$ cannot be connected by a path that remains in K.

Exercise 12 Let M be a differential manifold, and let $f : M \to M$ be a C^1 diffeomorphism with a hyperbolic fixed point p. Suppose that D_pf has a unique eigenvalue of absolute value less than 1 and that this eigenvalue is real and positive. Show that $W^{ss}(p) \setminus \{p\}$ has two connected components.

9.6 Comments

The perturbation f studied in this chapter is C^1 and has Lipschitz derivative. We could have constructed a C^∞ map by taking a "bump" function of the type

$$k(r) = \exp\Big(-\frac{1}{1-r^2}\Big)\mathbf{1}_{[-1,1]}(r).$$

From a numerical point of view, a polynomial "bump" seems preferable.
Here are three algorithms that allow us to visualize the compact set K.

- We choose a point x arbitrarily and iterate it a million times using the map f^{-1}. If the point is not the origin, its trajectory will converge to the attractor K. The transformation f^{-1} restricted to K is transitive. For most x, the trajectory should therefore converge to all points of the attractor. This is what is seen in practice. This method is the fastest one from a numerical point of view.
- The origin is an attracting fixed point for the map f and K is the complement of its basin of attraction. To visualize this basin, we fix a small disk with center the origin, and then color the points of the plane as a function of the number of iterations needed to reach this disk. In practice, most points reach the disk in less than 70 steps. We could, for example, color all points needing more than twenty iterations in black, which would allow us to represent a small neighborhood of K.
- We can show that the periodic points of f are dense in K. The set of periodic points with period less than n, for n sufficiently large, therefore gives a good approximation of K. Calculating the periodic points turns out to be very costly numerically, so this method is seldom recommended.

We can describe the dynamics of f restricted to K in a more precise way. On K, the transformation f is semiconjugate to a topologically mixing shift of finite type. R. F. Williams (1974) has shown that it is conjugate to a shift on a generalized solenoid. The behavior of f is therefore highly unpredictable.
The compact set K is locally homeomorphic to the product of a segment and a Cantor set. We can verify this in the neighborhood of p by showing that the intersection of K and the local unstable manifold of p has empty interior in K. To prove it in the neighborhood of every point x of K, we must study in detail the structure of the stable manifolds $W^{ss}(x)$ and show that they are all immersed submanifolds of dimension 1.
The points of $W^{ss}(p)$ and of $W^{ss}(-p)$ make up the accessible boundary of U. They are the only points of ∂U that are the endpoints of a curve $\gamma : [0,1] \to \mathbf{T}^2$ contained in U for $t \in [0,1)$. This notion of accessible boundary no doubt corresponds better to the intuitive idea one can have of the boundary of a set.
A DA diffeomorphism is an example of an Axiom A diffeomorphism: its periodic points form a dense subset of the nonwandering set $\{0\} \cup K$; when restricted to the latter, the tangent space can be decomposed into the sum of two invariant subbundles, respectively contracted and dilated by the differential of the map.

The nonwandering set of an Axiom A diffeomorphism decomposes into a finite number of invariant compact sets, restricted to which the transformation is transitive. This can be proved by studying the stable and unstable manifolds of the periodic points, as was done for the DA diffeomorphism.