Integrais duplas e mudança de variáveis:

Calcule $\iint_R \sqrt{4-x^2-y^2}\,dx\,dy$, onde $R=\{(x,y)/x\geq 0 \text{ e } 0\leq x^2+y^2\leq 4\}$.

Resp: Com coordenadas polares, $I=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^2 \sqrt{4-r^2}\, r\,dr\,d\theta = \left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta\right)\cdot\left(\int_0^2 r\sqrt{4-r^2}\,dr\right)$

$$= [\theta]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left[-\frac{1}{2}\,\frac{2}{3}\,(4-r^2)^{3/2}\right]_0^2 = \left(\frac{\pi}{2}+\frac{\pi}{2}\right)\left(-\frac{1}{3}(0-4^{3/2})\right)=\frac{8\pi}{3}.$$

Calcule $\iint_R y e^x\,dx\,dy$ onde $R=\{(x,y)/\ x\geq 0, y\geq 0 \text{ e } 0\leq x^2+y^2\leq 25\}$.

Resp: $\iint_R y e^x\,dx\,dy = \int_0^{\pi/2}\int_0^5 (r\,\mathrm{sen}\,\theta)\, e^{r\cos\theta}\, r\,dr\,d\theta = \int_0^5\int_0^{\pi/2} r^2\,\mathrm{sen}\,\theta\, e^{r\cos\theta}\,d\theta\,dr.$

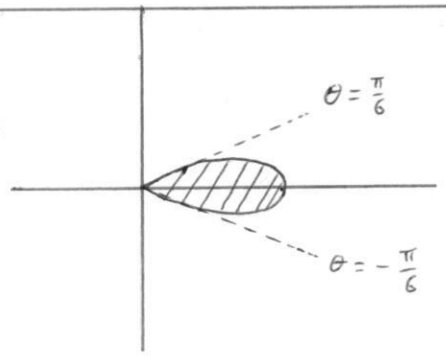
Agora: $\int_0^{\pi/2} r^2\,\mathrm{sen}\,\theta\, e^{r\cos\theta}\,d\theta$: seja $u=r\cos\theta$, $du=-r\,\mathrm{sen}\,\theta\,d\theta$

$$=\int_{u=r}^{u=0} -re^u\,du = -r\left(e^0-e^r\right)=re^r-r.$$

Depois: $\int_0^5\int_0^{\pi/2} r^2\,\mathrm{sen}\,\theta\, e^{r\cos\theta}\,d\theta\,dr = \int_0^5 (re^r-r)\,dr = \Big[\underbrace{re^r-r}_{\text{por integração por partes}} -\frac{1}{2}r^2\Big]_0^5 = 4e^5-\frac{23}{2}.$

Calcule a área da região $D=\{(r,\theta)/\ -\frac{\pi}{6}\leq\theta\leq\frac{\pi}{6},\ 0\leq r\leq \cos 3\theta\}$.

Região D :



$$\text{Área } D=\iint_D dx\,dy = \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}}\int_0^{\cos 3\theta} r\,dr\,d\theta$$

$$=\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}}\left[\frac{1}{2}r^2\right]_{r=0}^{r=\cos 3\theta} d\theta$$

$$=\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \frac{1}{2}\cos^2 3\theta\,d\theta$$

$$=2\int_0^{\pi/6}\frac{1}{2}\left(\frac{1+\cos 6\theta}{2}\right)d\theta$$

$$=\frac{1}{2}\left[\theta+\frac{1}{6}\,\mathrm{sen}\,6\theta\right]_0^{\pi/6}$$

$$=\frac{\pi}{12}.$$

Integrais duplas, triplas e cálculo de volumes

Volume da esfera de raio $r = a$

Resp: pela simetria, $V = 2 \iint_{x^2+y^2 \leq a^2} \sqrt{a^2-x^2-y^2}\, dx\, dy = 2\int_0^{2\pi}\int_0^a \sqrt{a^2-r^2}\, r\, dr\, d\theta$

$$= 2\left(\int_0^{2\pi} d\theta\right)\int_0^a r\sqrt{a^2-r^2}\, dr$$

$$= 2\, [\theta]_0^{2\pi} \left[-\frac{1}{3}(a^2-r^2)^{3/2}\right]_0^a = 2\,(2\pi)\left(0+\frac{1}{3}a^3\right) = \frac{4}{3}\pi a^3.$$

Volume da região entre a esfera de raio 1 e o cone $z = \sqrt{x^2+y^2}$.

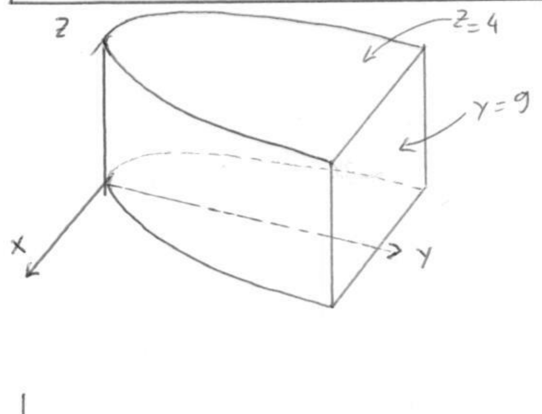
Resp: O cone e a esfera têm interseção descrita por $\left\{\begin{array}{l} x^2+y^2+z^2=1 \\ \text{e} \\ z=\sqrt{x^2+y^2} \end{array}\right\}$ $\Rightarrow x^2+y^2+\left(\sqrt{x^2+y^2}\right)^2 = 1$ $\Rightarrow x^2+y^2 = \frac{1}{2}$.

Então $V = \iint_{x^2+y^2 \leq 1/2} \left(\sqrt{1-x^2-y^2} - \sqrt{x^2+y^2}\right) dx\, dy = \int_0^{2\pi}\int_0^{1/\sqrt{2}} \left(\sqrt{1-r^2} - r\right) r\, dr\, d\theta$

$$= \left(\int_0^{2\pi} d\theta\right)\int_0^{1/\sqrt{2}} \left(r\sqrt{1-r^2} - r^2\right) dr = [\theta]_0^{2\pi}\left[-\frac{1}{3}(1-r^2)^{3/2} - \frac{1}{3}r^3\right]_0^{1/\sqrt{2}}$$

$$= 2\pi\left(-\frac{1}{3}\right)\left(\frac{1}{\sqrt{2}}-1\right) = \frac{\pi}{3}\left(2-\sqrt{2}\right).$$

Volume da região $E = \{(x,y,z) \,/\, x \in [-3,3],\ 9 \leq y \leq x^2,\ z \in [0,4]\}$, e esboço de E.



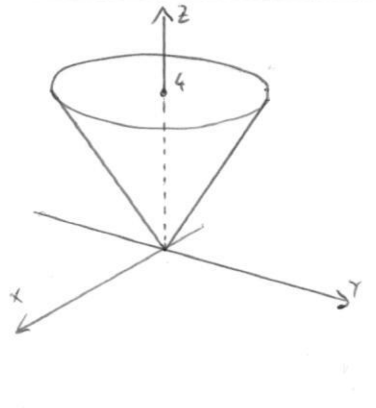
$$V = \int_{-3}^{3}\int_{x^2}^{9}\int_0^4 dz\, dy\, dx = 4\int_{-3}^{3}\int_{x^2}^{9} dy\, dx$$

$$= 4\int_{-3}^{3}(9-x^2)\, dx$$

$$= 4\left[9x - \frac{1}{3}x^3\right]_{-3}^{3} = 4(27-9+27-9)$$

$$= 144.$$

Volume da região entre $z = \sqrt{x^2+y^2}$ e $z = 4$ e esboço dessa região



Com coordenadas cilíndricas:

$$V = \int_0^4\int_0^{2\pi}\int_r^4 r\, dz\, d\theta\, dr = \int_0^4\int_0^{2\pi}[rz]_{z=r}^{z=4}\, d\theta\, dr = \int_0^4\int_0^{2\pi} r(4-r)\, d\theta\, dr$$

$$= \int_0^4 (4r-r^2)\, dr \int_0^{2\pi} d\theta = \left[2r^2 - \frac{1}{3}r^3\right]_0^4 [\theta]_0^{2\pi} = \left(32 - \frac{64}{3}\right)(2\pi)$$

$$= \frac{64\pi}{3}.$$

Integrais triplas

Seja E a região limitada pelo parabolóide $z = 1 + x^2 + y^2$, pelo cilindro $r = \sqrt{5}$ e pelo plano (xy). Calcule $\iiint_E e^z \, dx\,dy\,dz$.

Resp: temos: $E = \{(r,\theta,z) / \ 0 \leq \theta \leq 2\pi, \ 0 \leq r \leq \sqrt{5}, \ 0 \leq z \leq 1 + r^2\}$.

Então $I = \iiint_E e^z \, dx\,dy\,dz = \int_0^{2\pi} \int_0^{\sqrt{5}} \int_0^{1+r^2} e^z \, r \, dz \, dr \, d\theta = \int_0^{2\pi} \int_0^{\sqrt{5}} r \left[e^z\right]_{z=0}^{z=1+r^2} dr\,d\theta$.

$$= \int_0^{2\pi} \int_0^{\sqrt{5}} r\left(e^{1+r^2} - 1\right) dr\,d\theta = \int_0^{2\pi} d\theta \int_0^{\sqrt{5}} \left(r e^{1+r^2} - r\right) dr = 2\pi \left[\frac{1}{2} e^{1+r^2} - \frac{1}{2} r^2\right]_0^{\sqrt{5}}$$

$$= \pi \left(e^6 - e - 5\right).$$

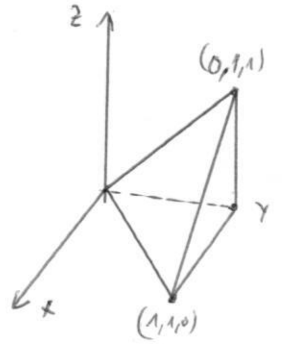
Seja $E = \{(r,\theta,z) / \ 0 \leq \theta \leq 2\pi, \ 0 \leq r \leq 1, \ 0 \leq z \leq 2r\}$. Calcule $\iiint_E x^2 \, dx\,dy\,dz = I$.

Resp: $I = \int_0^{2\pi} \int_0^1 \int_0^{2r} r^2 \cos^2\theta \, r \, dz \, dr \, d\theta = \int_0^{2\pi} \int_0^1 \left[r^3 \cos^2\theta\right]_{z=0}^{z=2r} dr\,d\theta$

$$= \int_0^{2\pi} \int_0^1 2r^4 \cos^2\theta \, dr\,d\theta = \int_0^{2\pi} \left[\frac{2}{5} r^5 \cos^2\theta\right]_{r=0}^{r=1} d\theta = \frac{2}{5} \int_0^{2\pi} \cos^2\theta \, d\theta$$

$$= \frac{2}{5} \int_0^{2\pi} \frac{1 + \cos 2\theta}{2} d\theta = \frac{1}{5} \left[\theta + \frac{1}{2} \operatorname{sen} 2\theta\right]_0^{2\pi} = \frac{2\pi}{5}.$$

Calcule $\int_0^1 \int_0^y \int_0^{y-z} xz \, dx\,dy\,dy$ e desenhe a região de integração.

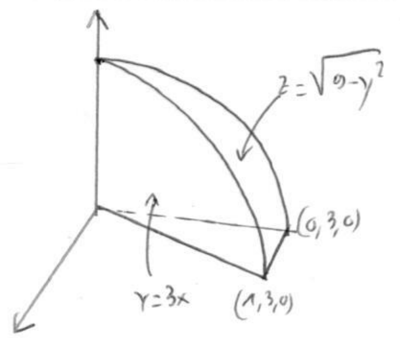


$$\int_0^1 \int_0^y \int_0^{y-z} xz \, dx\,dz\,dy = \int_0^1 \int_0^y \left(\frac{1}{2}\right)(y-z)^2 z \, dz\,dy$$

$$= \frac{1}{2} \int_0^1 \left[\frac{1}{2} y^2 z^2 - \frac{2}{3} y z^3 + \frac{1}{4} z^4\right]_{z=0}^{z=y} dy$$

$$= \frac{1}{24} \int_0^1 y^4 \, dy = \frac{1}{24} \left(\frac{1}{5} y^5\right]_0^1 = \frac{1}{120}.$$

Calcule $\int_0^1 \int_{3x}^3 \int_0^{\sqrt{9-y^2}} z \, dz\,dy\,dx$ e desenhe a região de integração.



Resp: $I = \int_0^1 \int_{3x}^3 \int_0^{\sqrt{9-y^2}} z \, dz\,dy\,dx$

$$= \int_0^1 \int_{3x}^3 \frac{1}{2} (9 - y^2) \, dy\,dx = \int_0^1 \left[\frac{9}{2} y - \frac{1}{6} y^3\right]_{y=3x}^{y=3} dx$$

$$= \int_0^1 \left[9 - \frac{27}{2} x + \frac{9}{2} x^3\right] dx$$

$$= \left[9x - \frac{27}{4} x^2 + \frac{9}{8} x^4\right]_0^1 = \frac{27}{8}.$$