Integrais duplas e mudança de variáveis:

Calcule $\iint_R \sqrt{4-x^2-y^2}\,dx\,dy$, onde $R=\left\{(x,y)/x\geq 0 \text{ e } 0\leq x^2+y^2\leq 4\right\}$.

Resp: Com coordenadas polares, $I=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^2 \sqrt{4-r^2}\, r\,dr\,d\theta = \left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta\right)\cdot\left(\int_0^2 r\sqrt{4-r^2}\,dr\right)$

$$= [\theta]_{-\frac{\pi}{2}}^{\pi/2}\left[-\frac{1}{2}\,\frac{2}{3}\,(4-r^2)^{3/2}\right]_0^2 = \left(\frac{\pi}{2}+\frac{\pi}{2}\right)\left(-\frac{1}{3}\left(0-4^{3/2}\right)\right)=\frac{8\pi}{3}.$$

Calcule $\iint_R ye^x\,dx\,dy$ onde $R=\left\{(x,y)/ x\geq 0, y\geq 0 \text{ e } 0\leq x^2+y^2\leq 25\right\}$.

Resp: $\iint_R ye^x\,dx\,dy = \int_0^{\pi/2}\int_0^5 (r\,\text{sen}\,\theta)\,e^{r\cos\theta}\,r\,dr\,d\theta = \int_0^5\int_0^{\pi/2} r^2\,\text{sen}\,\theta\,e^{r\cos\theta}\,d\theta\,dr.$

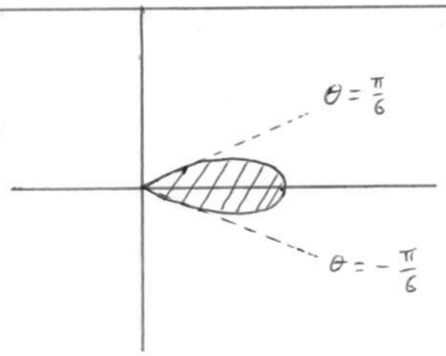
Agora: $\int_0^{\pi/2} r^2\,\text{sen}\,\theta\,e^{r\cos\theta}\,d\theta$: seja $u=r\cos\theta$, $du=-r\,\text{sen}\,\theta\,d\theta$

$$=\int_{u=r}^{u=0} -re^u\,du = -r\left(e^0-e^r\right) = re^r-r.$$

Depois: $\int_0^5\int_0^{\pi/2} r^2\,\text{sen}\,\theta\,e^{r\cos\theta}\,d\theta\,dr = \int_0^5 (re^r-r)\,dr = \Big[\underbrace{re^r-r}_{\text{por integração por partes}}\;-\frac{1}{2}r^2\Big]_0^5 = 4e^5-\frac{23}{2}.$

Calcule a área da região $D=\left\{(r,\theta)/ -\frac{\pi}{6}\leq\theta\leq\frac{\pi}{6},\ 0\leq r\leq\cos 3\theta\right\}$.

Região D :



$$\text{Área } D=\iint_D dx\,dy=\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}}\int_0^{\cos 3\theta} r\,dr\,d\theta$$

$$=\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}}\left[\frac{1}{2}r^2\right]_{r=0}^{r=\cos 3\theta} d\theta$$

$$=\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}}\frac{1}{2}\cos^2 3\theta\,d\theta$$

$$=2\int_0^{\pi/6}\frac{1}{2}\left(\frac{1+\cos 6\theta}{2}\right)d\theta$$

$$=\frac{1}{2}\left[\theta+\frac{1}{6}\,\text{sen}\,6\theta\right]_0^{\pi/6}$$

$$=\frac{\pi}{12}.$$

Integrais duplas, e cálculo de volumes

Volume da esfera de raio $r = a$

Resp: pela simetria, $V = 2 \iint_{x^2+y^2 \leq a^2} \sqrt{a^2-x^2-y^2}\, dx\, dy = 2\int_0^{2\pi}\int_0^a \sqrt{a^2-r^2}\, r\, dr\, d\theta$

$$= 2\left(\int_0^{2\pi} d\theta\right)\int_0^a r\sqrt{a^2-r^2}\, dr$$

$$= 2\,[\theta]_0^{2\pi}\left[-\frac{1}{3}(a^2-r^2)^{3/2}\right]_0^a = 2\,(2\pi)\left(0+\frac{1}{3}a^3\right) = \frac{4}{3}\pi a^3.$$

Volume da região entre a esfera de raio 1 e o cone $z = \sqrt{x^2+y^2}$.

Resp: O cone e a esfera têm interseção descrita por $\left\{\begin{array}{c} x^2+y^2+z^2=1 \\ e \\ z=\sqrt{x^2+y^2}\end{array}\right\}$ $\Rightarrow x^2+y^2+\left(\sqrt{x^2+y^2}\right)^2 = 1$ $\Rightarrow x^2+y^2 = \frac{1}{2}$.

Então $V = \iint_{x^2+y^2 \leq 1/2} \left(\sqrt{1-x^2-y^2} - \sqrt{x^2+y^2}\right) dx\, dy = \int_0^{2\pi}\int_0^{1/\sqrt{2}} \left(\sqrt{1-r^2} - r\right) r\, dr\, d\theta$

$$= \left(\int_0^{2\pi} d\theta\right)\int_0^{1/\sqrt{2}} \left(r\sqrt{1-r^2} - r^2\right) dr = [\theta]_0^{2\pi}\left[-\frac{1}{3}(1-r^2)^{3/2} - \frac{1}{3}r^3\right]_0^{1/\sqrt{2}}$$

$$= 2\pi\left(-\frac{1}{3}\right)\left(\frac{1}{\sqrt{2}} - 1\right) = \frac{\pi}{3}\left(2-\sqrt{2}\right).$$